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INVERSE SCATTERING

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Final Report to
Contract N00019-72-C-0462

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DEPARTMENT OF THE NAVY
NAVAL AIR SYSTEMS COMMAND
WASHINGTON, D. C. 20360

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FOREWORD

This report is divided into two completely separate and distinct parts. In Part I, *Physical Optics Inverse Scattering*, a physical optics approximation to the monostatic electromagnetic inverse scattering problem is developed for stationary perfectly conducting scatterers. In Part II, *Exact Inverse Scattering*, a rigorous and exact solution to the general inverse problem is developed, which is applicable to the monostatic-bistatic electromagnetic inverse scattering problem for moving scatterers of arbitrary conductivity. This division into two distinct parts is necessitated by the totally different approaches taken to the problem; although the results of Part I are reaffirmed by the incidental first-order approximations developed in Part II.

ABSTRACT

PART I: A three-dimensional electromagnetic Inverse Scattering Identity, based on the Physical Optics approximation, is developed for the monostatic scattered far field cross section of perfect conductors. Uniqueness of this inverse scattering identity is proven. This identity requires complete scattering information for all frequencies and aspect angles. A non-singular integral equation is developed for the arbitrary case of incomplete frequency and/or aspect angle scattering information. A general closed form solution to this integral equation is developed, which yields the shape of a scatterer from such incomplete information. A specific practical radar solution is presented. The resolution of this solution is developed; yielding short-pulse target resolution radar system parameter equations. Results of the three-dimensional numerical reconstruction of a sphere and a cylinder from a variety of aspect angle and frequency band limited cross section data are presented. The merits of this solution over the conventional synthetic aperture radar imaging technique are discussed.

PART II: The general inverse scattering and radiation problem associated with the three-dimensional inhomogeneous scalar field Helmholtz wave equation is formulated as a Fredholm integro-differential equation of the second kind. The far-field inverse integro-differential equation is solved in closed form with the aid of a single resolvent integral operator, which can be readily evaluated numerically with the aid of the fast Fourier transform algorithm. The inverse integro-differential equation and its solution are then generalized to the reduced vector wave equation resulting from Maxwell's equations. A formal statement of the inverse problem is presented. It is shown that the first order Neumann series solution of the inverse integro-differential equation as well as the first order term of its exact solution represent the physical optics approximation and the equations governing synthetic microwave holography. The (analogous) four-dimensional inverse integro-differential equation and its closed form solution, applicable to Doppler-contaminated fields radiated by moving scatterers, is developed.

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SECTION I

INTRODUCTION

The Direct Scattering problem, whether electromagnetic, acoustic, particle, or quantum mechanical, is defined as the problem of predicting the scattered quantities, given the incident quantities, the relevant description of the scatterer, and the appropriate laws governing the interaction. The back-scattering and forward-scattering problems are the special monostatic case and the special bistatic case of scattering in the direction of incidence respectively.

The Inverse Scattering problem is defined as the problem of determining the relevant quantities describing the scatterer, given the incident and scattered quantities, and the appropriate laws governing the interaction. The Uniqueness and Well-behavedness problems must also be taken as an integral part of the inverse scattering problem. The Well-behavedness problem is the problem of determining the degree of smoothness and continuity with which the so-called output data varies with respect to the so-called input data. The uniqueness problem must also deal with the question of incomplete input data.

The Electromagnetic Inverse Scattering problem is thus defined as the problem of determining the size, shape and electromagnetic properties distributions (conductivity, susceptibility, and permeability distributions) of a scatterer, given the incident and scattered electromagnetic fields, and the electromagnetic field equations (Maxwell's equations and the appropriate wave equations; the constitutive equations and their coefficients are taken as part of the electromagnetic properties of the scatterer); and the determination of whether this problem is uniquely solvable for incomplete input data, i.e., the various permutations of incomplete bistatic aspect angles, incomplete monostatic aspect angles, incomplete frequency, monochromatic data only, incomplete polarization matrix, amplitude (power cross-section) data only, and scattered far field data only. Also of interest is the inverse scattering problem for which some a priori information about the scatterer exists; e.g., that the conductivity of the scatterer is infinite (perfect scatterer), that the geometry of the scatterer is of axial symmetry, etc.

For a physically more sophisticated and mathematically more rigorous definition of the general inverse problem (including the inverse radiation problem), the reader is referred to part II, sections IX and X.5 of this report.

The subject matter of this part I is restricted to the monostatic far field special case; a restriction being characteristically inherent to the radar application. For mathematical reasons that will become evident to the reader, the further special case of a priori knowledge of the scatterer being a perfect conductor is treated. For both of these cases, special attention is given to the further special case of the finite three-dimensional aperture; i.e., the case for which monostatic scattering data is available for a limited and incomplete frequency and aspect angle domain. From a practical short pulse target resolution radar point of view, this special case is of fundamental and primary interest.

The approach taken and the results obtained can be summarized as follows. In Section II, the physical optics approximation is taken as the basis for the direct scattering theory, and its validity for, and consistency with, short pulse target resolution radar concepts is discussed briefly. In Section III, with the aid of this approximation, a basic Inverse Scattering Identity is then developed, that states that the characteristic function (in three-dimensional space) of a scatterer is related to the field cross-section (in three-dimensional k-space) by a three-dimensional Fourier Transform. Uniqueness of the solution for finite sized scatterers is established. In Section IV, a general, non-singular, inverse scattering integral equation is developed, solutions to which permit the determination of an appropriate maximum of information about a scatterer from incomplete scattering data. The use of the Physical Optics approximation is further justified. In Section

V, a closed form solution to this integral equation, valid for any arbitrarily shaped finite k-space aperture (of incomplete frequency and aspect angles scattering data), is developed. The full details of a practical frequency and aspect angles band limited right rectangular quasi-conic section aperture are then developed as an example. In Section VI, the spatial resolution of the solution obtained is developed, yielding a set of short pulse target resolution radar system parameters. In Section VII, the special cases of two- and one-dimensional inverse scattering, the special case of certain given simple geometrical shapes of the aperture, and the special case of a priori knowledge of certain symmetries possessed by the scatterer, are discussed. The special case of a priori knowledge of the scatterer being a surface of revolution about some axis is treated in detail. In section VIII, results of the three-dimensional numerical reconstruction of a sphere from a variety of aspect angle and frequency band limited cross section data, obtained numericaly by Mie's exact solution for the direct scattering by a sphere, are presented. Similar results for a cylinder are also presented. In section IX, plans for future numerico-experimental work are outlined, and the extension of the monostatic physical optics inverse scattering solution for stationary perfectly conducting scatterers to a generalized monostatic-bistatic inverse scattering solution for moving scatterers of arbitrary (unknown) conductivity is briefly discussed. In the concluding section IX, conclusions are drawn about the merits of the presented solutions over the conventional synthetic aperture radar imaging technique.

SECTION II

THE PHYSICAL OPTICS CROSS SECTION

The scattered magnetic field H^s in terms of the induced (by the incident field) surface current density K on the surface of a perfect conductor is given by [1]

$$H^s = \oint_s \nabla\phi \times K \, ds \quad , \tag{1}$$

where the Green's function ϕ and its gradient are given by

$$\phi = \frac{e^{ikr}}{4\pi r} \tag{2}$$

$$\nabla\phi = \frac{ikr-1}{r^2} \, r \, \phi \, . \tag{3}$$

The Physical Optics Approximation [2] for the induced surface current density in terms of the incident magnetic field H^i is

$$K = \begin{cases} 2n\times H^i, & \text{on the "illuminated" segment of s,} \\ 0 \quad , & \text{on the "shadow" segment of s.} \end{cases} \tag{4}$$

Thus, by (1), (3), and (4), the scattered magnetic far field H^f, in terms of the wave number propagation vector k^s of the scattered far field, is

$$H^f = -2ik^s \times \int_{k^i \cdot n<0} \phi \; n\times H^i \, ds \, . \tag{5}$$

If the incident field H^i is taken as a plane wave of the form

$$H^i = I\ e^{ik^i \cdot x} \tag{6}$$

in the vicinity of the scatterer, where k^i is the wave number propagation vector of the incident field, and the range and phase normalized (in the coordinate system in which the scatterer is described) scattered far field is taken as

$$H^f = \frac{1}{\sqrt{4\pi r^2}} S\ e^{ik^s \cdot x} , \tag{7}$$

then for the case of

$$k^s = -k^i \equiv k , \tag{8}$$

i.e., the monostatic case, (5) reduces with the aid of (2), (8), and the transversality of the incident field to

$$S = \frac{i}{\sqrt{\pi}} \int_{k \cdot n > 0} e^{-2ik\cdot x}\ k \cdot ds\ I . \tag{9}$$

Consistent with the conventional definition [3] of the power cross section σ and the field cross section ρ

$$S = \rho\ I \tag{10}$$

$$\sigma = \rho\rho^* \equiv 4\pi r^2 \frac{|H^s|^2}{|H^i|^2} , \tag{11}$$

equation (9) yields for the physical optics field cross section the well-known expression [4]

$$\rho = \frac{i}{\sqrt{\pi}} \int_{k \cdot n > 0} e^{-2ik\cdot x}\ k \cdot ds . \tag{12}$$

For a short-pulse target resolution radar system to be effective, its pulse length must be short compared to the target size; furthermore, since the fractional bandwidth of such a pulse is limited by practical considerations to much less than unity, it follows that the largest wave length in the spectrum of the transmitted pulse must be very short indeed compared to the target size. The physical optics approximation (12) is thus a good model for the direct scattering theory for such a short-pulse target resolution radar system. For a detailed discussion of the physical meaning and implications of the physical optics approximation, i.e., its being a total first order local scattering theory, consistent with short-pulse radar concepts, the reader is referred to an earlier work of this author [5].

SECTION III

THE INVERSE SCATTERING IDENTITY

Introducing the variable κ defined as

$$\kappa \equiv 2k \tag{13}$$

yields for (12)

$$\rho(\kappa) = \frac{i}{\sqrt{4\pi}} \int_{\kappa\cdot n>0} e^{-i\kappa\cdot x}\, \kappa\cdot ds \ . \tag{14}$$

Thus

$$\rho^*(-\kappa) = \frac{i}{\sqrt{4\pi}} \int_{\kappa\cdot n<0} e^{-i\kappa\cdot x}\, \kappa\cdot ds \ , \tag{15}$$

and

$$\rho(\kappa) + \rho^*(-\kappa) = \frac{i}{\sqrt{4\pi}} \int_{\kappa\cdot n>0} e^{-i\kappa\cdot x}\, \kappa\cdot ds + \frac{i}{\sqrt{4\pi}} \int_{\kappa\cdot n<0} e^{-i\kappa\cdot x}\, \kappa\cdot ds \tag{16}$$

$$= \frac{i}{\sqrt{4\pi}} \oint_{s} e^{-i\kappa\cdot x}\, \kappa\cdot ds \ . \tag{17}$$

Since the integrand of (17) is continuous and differentiable on s and in v banded by s, it follows by Gauss' theorem that

$$\rho(\kappa) + \rho^*(-\kappa) = \frac{i}{\sqrt{4\pi}} \int_v \nabla \cdot (\kappa e^{-i\kappa \cdot X})\, dv \tag{18}$$

$$= \frac{\kappa^2}{\sqrt{4\pi}} \int_v e^{-i\kappa \cdot X}\, dv \; . \tag{19}$$

Introducing the quantity $\Gamma(\kappa)$ defined by

$$\Gamma(\kappa) \equiv \sqrt{4\pi}\, \frac{\rho(\kappa) + \rho^*(-\kappa)}{\kappa^2} \; , \tag{20}$$

yields for (19)

$$\Gamma(\kappa) = \int_v e^{-i\kappa \cdot X}\, dv \; . \tag{21}$$

Defining the characteristic function $\gamma(X)$ of the scatterer by

$$\gamma(X) \equiv \begin{cases} 1 \; , & X \text{ in } v \; , \\ 0 \; , & X \text{ not in } v \; , \end{cases} \tag{22}$$

permits the reformulation of (21) as the three-dimensional Fourier integral

$$\Gamma(\kappa) = \int_{-\infty}^{\infty} e^{-i\kappa \cdot X}\, \gamma(X)\, d^3x \; . \tag{23}$$

If the volume v of the scatterer is finite, then by (22)

$$\int_{-\infty}^{\infty} |\gamma(X)|\, d^3x = v$$

$$< \infty \; ; \tag{24}$$

it thus follows from (23) that for finite sized scatterers the three-dimensional inverse Fourier transform of $\Gamma(\kappa)$ exists uniquely; i.e.

$$\boxed{\gamma(x) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\kappa\cdot x}\, \Gamma(\kappa)\, d^3\kappa\ .} \tag{25}$$

With the aid of (20), (25) can be reformulated as follows

$$\gamma(x) = \frac{1}{4\pi^{5/2}} \left[\int_{-\infty}^{\infty} e^{i\kappa\cdot x}\, \frac{\rho(\kappa)}{\kappa^2}\, d^3\kappa + \int_{-\infty}^{\infty} e^{i\kappa\cdot x}\, \frac{\rho^*(-\kappa)}{\kappa^2}\, d^3\kappa \right]\ ; \tag{26}$$

which, after replacing κ by $-\kappa$ in the second integral of (26) yields

$$\gamma(x) = \frac{1}{4\pi^{5/2}} \left[\int_{-\infty}^{\infty} e^{i\kappa\cdot x}\, \frac{\rho(\kappa)}{\kappa^2}\, d^3\kappa + \int_{-\infty}^{\infty} e^{-i\kappa\cdot x}\, \frac{\rho^*(\kappa)}{\kappa^2}\, d^3\kappa \right] \tag{27}$$

$$= \left[\frac{1}{4\pi^{5/2}} \int_{-\infty}^{\infty} e^{i\kappa\cdot x}\, \frac{\rho(\kappa)}{\kappa^2}\, d^3\kappa \right] + \left[\frac{1}{4\pi^{5/2}} \int_{-\infty}^{\infty} e^{i\kappa\cdot x}\, \frac{\rho(\kappa)}{\kappa^2}\, d^3\kappa \right]^* \tag{28}$$

$$= \frac{1}{2\pi^{5/2}}\, \mathcal{R}e \int_{-\infty}^{\infty} e^{i\kappa\cdot x}\, \frac{\rho(\kappa)}{\kappa^2}\, d^3\kappa\ . \tag{29}$$

Both inverse scattering identities (25) and (29) clearly require complete scattering information; namely, knowledge of $\rho(\kappa)$ over all κ-space (i.e., all frequencies and all aspect angles).

SECTION IV

THE FINITE APERTURE INTEGRAL EQUATION

In practice $\rho(\kappa)$ is known (measurable) only for an incomplete finite portion of the complete κ-space; namely a κ-space aperture consisting of a limited (finite) frequency band and a limited aspect angles band. Furthermore (23) is valid only in the Physical Optics regime (wave length short compared to the overall size of the scatterer), and hence (25) must either include fictitious (Physical Optics scattering data in the Rayleigh regime, which is physically not realizable) low frequency scattering data $\Gamma(\kappa)$, or no such data at all. It is thus to the problem of determining what can be deduced about a scatterer (i.e., $\gamma(X)$) from such limited high frequency finite aperture data that the ensuing sections are addressed.

Let $A(\kappa)$ be an *aperture function* defined as

$$A(\kappa) = C(\kappa)\, W(\kappa)\ , \tag{30}$$

where $C(\kappa)$ is a *characteristic aperture function* defined as

$$C(\kappa) = \begin{cases} 1\ , & \text{for } \kappa \text{ for which } \Gamma(\kappa) \text{ is known,} \\ 0\ , & \text{for } \kappa \text{ for which } \Gamma(\kappa) \text{ is unknown,} \end{cases} \tag{31}$$

and where $W(\kappa)$ is any appropriately chosen (in general non-zero) *aperture weighting function,* subject to the conditions

$$\int_{-\infty}^{\infty} |A(\kappa)|\ d^3\kappa < \infty\ . \tag{32}$$

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Thus, if the κ-space volume of the aperture is finite, and/or the aperture weighting function is appropriately chosen, then the three-dimensional inverse Fourier transforms of the aperture and characteristic aperture functions exist uniquely, i.e.

$$a(X) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\kappa \cdot X} A(\kappa)\, d^3\kappa \tag{33}$$

$$c(X) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\kappa \cdot X} C(\kappa)\, d^3\kappa \;. \tag{34}$$

Thus, by (25) and the three-dimensional convolution theorem for three-dimensional Fourier transforms, it follows that

$$a(X)*\gamma(X) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\kappa \cdot X} \Gamma(\kappa)\, A(\kappa)\, d^3\kappa \;, \tag{35}$$

which by (30) and (31) reduces to

$$a(X)*\gamma(X) = \frac{1}{(2\pi)^3} \int_{C} e^{i\kappa \cdot X} \Gamma(\kappa)\, W(\kappa)\, d^3\kappa \;, \tag{36}$$

where $\Gamma(\kappa)$ is clearly known in the domain of integration C: i.e., the aperture. The right-hand side of (36) can thus be taken as known; say the known function $g(X)$; i.e.

$$g(X) \equiv \frac{1}{(2\pi)^3} \int_{C} e^{i\kappa \cdot X} \Gamma(\kappa)\, W(\kappa)\, d^3\kappa \;. \tag{37}$$

The three-dimensional inverse scattering problem for a finite aperture thus reduces by (36) and (37) to the three-dimensional non-singular convolution integral equation (a Fredholm integral equation of the First Kind)

$$\boxed{\int a(\mathbf{X}-\mathbf{X}')\ \gamma(\mathbf{X}')\ d^3x' = g(\mathbf{X})\quad .} \tag{38}$$

This integral equation (38) can be solved numerically by a variety of existing techniques such as the matrix methods of Ritz-Galerkin [6], the associated Least Square Best Estimate method [7], and the associated moments method of Harrington [8], the Eigen-function expression method of Toraldo Di Francia [9], leading to so-called super-resolution, and the k-space method of this author [10], which also leads to super resolution. Several closed form solutions of (38) for apertures of specific geometry have been obtained by Lewis [11]; an alternate closed form solution of (38) for apertures of general arbitrary geometry is presented next.

SECTION V

A SOLUTION OF THE FINITE APERTURE INTEGRAL EQUATION

The solution of (38) for $\gamma(X)$ is greatly facilitated by the special properties of $\gamma(X)$ (i.e., a priori knowledge that $\gamma(X)$ is a characteristic function of the form (22)) and the possible judicious choice of the aperture function $F(\kappa)$.

Let the x_3-axis be chosen as passing through the (near) center of the aperture A (see fig. 1). Next, let the aperture function $W(\kappa)$ be chosen as

$$W(\kappa) = i\kappa_3 \tag{39}$$

It thus follows from (30), (32), (33), and (34), and again the three-dimensional convolution theorem for three-dimensional Fourier transforms that

$$a(X) = c(X)*\delta(x_1)\delta(x_2)\delta'(x_3) \ , \tag{40}$$

where $c(X)$ is known, i.e.

$$c(X) = \frac{1}{(2\pi)^3}\int_C e^{i\kappa\cdot X}\, d^3\kappa \ . \tag{41}$$

It thus follows from (38) that

$$c(X)*\delta(x_1)\delta(x_2)\delta'(x_3)*\gamma(X) = g(X) \ , \tag{42}$$

which reduces to

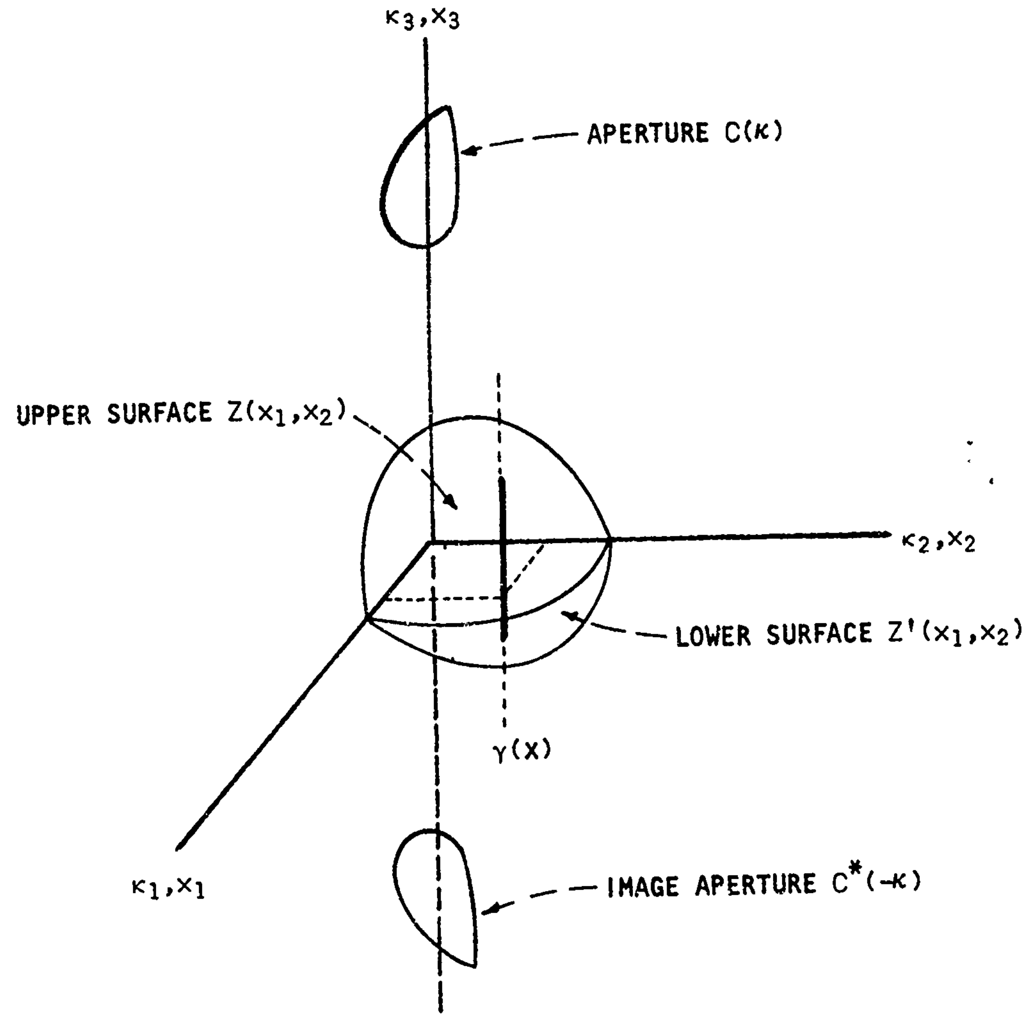

FIGURE 1

THE APERTURE AND SURFACE COORDINATE SYSTEM

$$c(X) * \frac{\partial\gamma}{\partial x_3} = g(X) \ . \tag{43}$$

Since $\gamma(X)$ is a characteristic function of the form (22); namely and particularly, that for fixed values of x_1 and x_2, $\gamma(x_3)$ is a dual step function in x_3 of unity magnitude and steps at the lower and upper surfaces, say $Z'(x_1,x_2)$ and $Z(x_1,x_2)$ respectively, of the scatterer (see fig. 1); it follows that

$$\frac{\partial\gamma}{\partial x_3} = \delta(x_3 - Z'(x_1,x_2)) - \delta(x_3 - Z(x_1,x_2)) \ , \tag{44}$$

which by (43) yields

$$c(X)*\delta(x_3 - Z'(x_1,x_2)) - c(X)*\delta(x_3 - Z(x_1,x_2)) = g(X) \ . \tag{45}$$

Examination of (20), (26), and (37), and symmetry and physical considerations (i.e., the implications of (4) and (12)), thus yields for the upper surface $Z(x_1,x_2)$ only

$$c(X)*\delta(x_3 - Z(x_1,x_2)) = - \frac{i}{4\pi^{5/2}} \int_C e^{i\kappa \cdot X} \left[\frac{\kappa_3 \ \rho(\kappa)}{\kappa^2} \right] d^3\kappa \ , \tag{46}$$

provided

$$W(\kappa) = W^*(-\kappa) \ , \tag{47}$$

which is assured by (39).

It is physically reasonable (by the implications of (4) and (12)) that information about the lower surface $Z'(x_1,x_2)$ should only be obtainable from scattering data from the lower image aperture $A^*(-\kappa)$. It is now evident that the introduction of the image aperture served the sole purpose of a mathematical artifice which permitted the application of Gauss' theorem to (12), yielding (25) and (29); and that knowledge of scattering data in this image aperture is not needed.

The three-dimensional convolution on the left-hand side of (46), say $\chi(\mathbf{X})$, reduces to

$$\chi(\mathbf{X}) \equiv c(\mathbf{X}) * \delta(x_3 - Z(x_1,x_2)) \tag{48}$$

$$= \iint_{-\infty}^{\infty} c(x_1 - x_1', x_2 - x_2', x_3 - Z(x_1', x_2'))\, dx_1'\, dx_2' \quad . \tag{49}$$

Thus

$$\boxed{\chi(\mathbf{X}) = -\frac{i}{4\pi^{5/2}} \int_C e^{i\boldsymbol{\kappa}\cdot\mathbf{X}} \left[\frac{\kappa_3\, \rho(\boldsymbol{\kappa})}{\kappa^2}\right] d^3\kappa \ ,} \tag{50}$$

where $\chi(\mathbf{X})$ is a resolution density function which is a measure of the location of the upper surface $Z(x_1,x_2)$. That $\chi(\mathbf{X})$ is indeed such a resolution density function can best be visualized by considering the limiting case of an infinite aperture function $A(\boldsymbol{\kappa})$ for which $c(\mathbf{X}) = \delta(\mathbf{X})$; for such an aperture, by (49)

$$\chi(\mathbf{X}) = \iint_{-\infty}^{\infty} \delta(x_1 - x_1')\ \delta(x_2 - x_2')\ \delta(x_3 - Z(x_1', x_2'))\ dx_1'\ dx_2' \tag{51}$$

$$= \delta(x_3 - Z(x_1,x_2)) \quad ; \tag{52}$$

whereas for a practical realistic aperture of finite $\boldsymbol{\kappa}$-space extent, the spatial extent of the non-vanishing portion of $c(\mathbf{X})$, and hence the non-vanishing portion of $\chi(\mathbf{X})$, is still small compared to the size of the scatterer (see Sect. VI). In fact, it is this resolution function $\chi(\mathbf{X})$ which determines the resolution of the solution (50); a resolution which can only be exceeded by the super-resolution method mentioned earlier.

A three-dimensional density plot of $|\chi(\mathbf{X})|$ thus represents the smeared geometrical image of the surface of the scatterer; the spatial extent of the smearing clearly being the spatial extent of $c(\mathbf{X})$, i.e., the resolution.

A Best Estimate of $Z(x_1,x_2)$ can alternatively be obtained by a variety of correlation (between (41) and (49)) methods, employing Fourier transform theory.

SECTION VI

THE RESOLUTION

The resolution in x-space is clearly the spatial extent of the nearly non-vanishing extent of $\chi(X)$. It thus follows from (30) through (38) and (49) that the resolution in any one direction in x-space is the reciprocal of the κ-spatial extent of the non-vanishing portion of the *Characteristic Aperture Function* $C(\kappa)$ in that same one direction.

The finite aperture inverse scattering solution (50) can clearly be reformulated in a variety of desired practical (radar) spherical coordinate systems. For the particular spherical coordinate system shown in fig. 2, i.e.

$$\kappa = \kappa \begin{pmatrix} \cos\xi & \sin\zeta \\ \sin\xi & \\ \cos\xi & \cos\zeta \end{pmatrix} \tag{53}$$

$$d^3\kappa = \kappa^2\, d\kappa \cos\xi\, d\xi\, d\zeta\ , \tag{54}$$

the inverse scattering solution (50) for the right rectangular quasi-conic section aperture shown becomes

$$\chi(X) = \frac{i}{4\pi^{5/2}} \int_{\kappa_1}^{\kappa_2} \int_{-\xi_o}^{\xi_o} e^{i\kappa x_2 \sin\xi} \int_{-\zeta_o}^{\zeta_o} e^{i\kappa \cos\xi\,(x_1 \sin\zeta + x_2 \cos\zeta)}\, \rho(\kappa,\xi,\zeta)$$

$$\times \cos\zeta\, d\zeta\, \cos^2\xi\, d\xi\, \kappa\, d\kappa\ , \tag{55}$$

where

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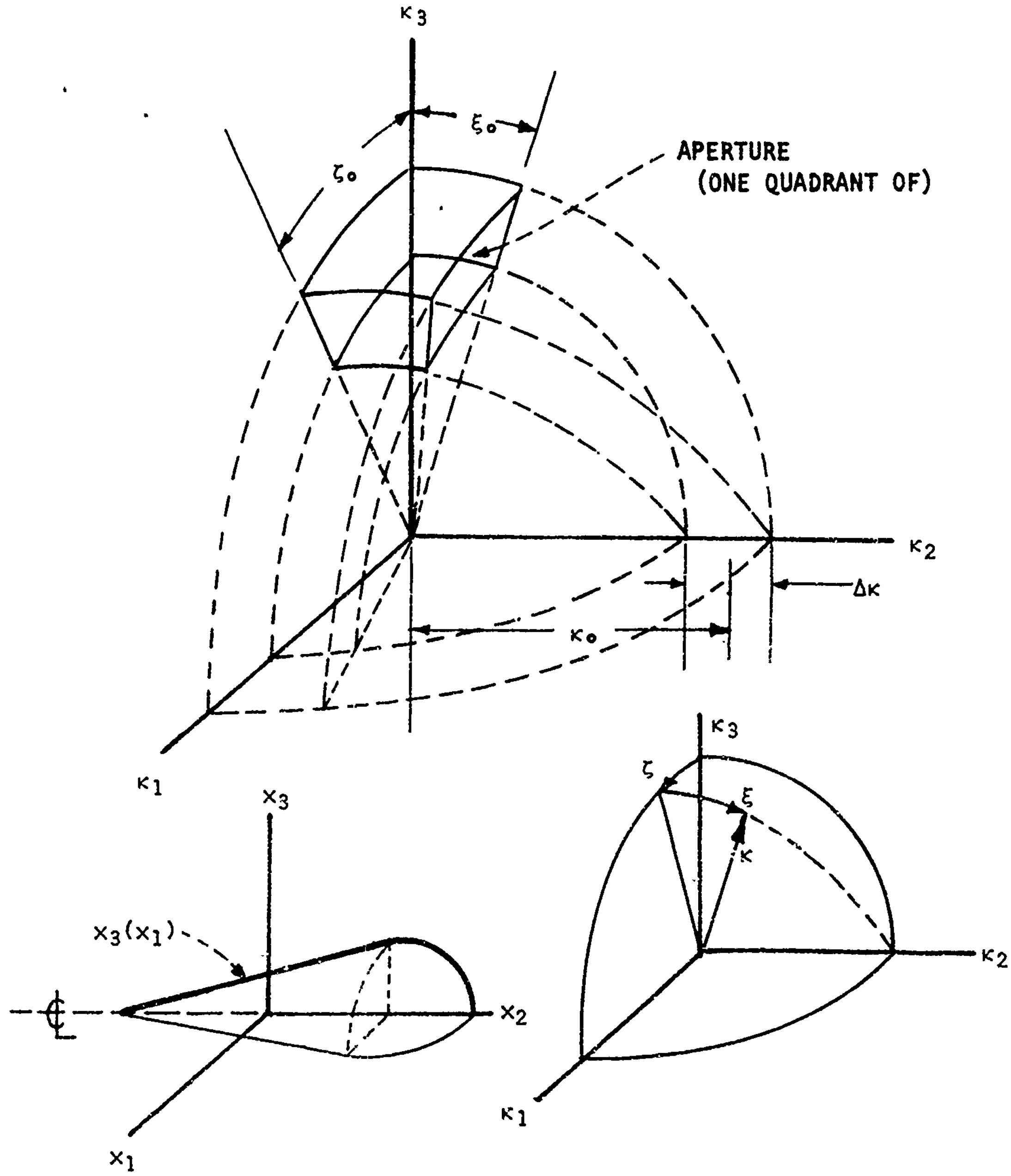

FIGURE 2

THE MIXED COORDINATE SYSTEM

$$\kappa_1 = \kappa_o - \tfrac{1}{2}\Delta\kappa \qquad (56)$$

$$\kappa_2 = \kappa_o + \tfrac{1}{2}\Delta\kappa \quad , \qquad (57)$$

where κ_o and $\Delta\kappa$ are the carrier and bandwidth of the transmitted spectrum respectively.

Examination of (41) and (49), after a similar reformulation into this same spherical coordinate system, readily reveals (after small angles approximations) the resolution in range and cross ranges to be

$$\Delta x_1 = \frac{1}{\kappa_o \ \Delta\zeta} \ ; \qquad \Delta\zeta \equiv 2\zeta_o \qquad (58)$$

$$\Delta x_2 = \frac{1}{\kappa_o \ \Delta\xi} \ ; \qquad \Delta\xi \equiv 2\xi_o \qquad (59)$$

$$\Delta x_3 = \frac{1}{\Delta\kappa} \ . \qquad (60)$$

This resolution can only be exceeded by the earlier mentioned techniques of super resolution.

Equations (58), (59), and (60) are thus the equations for the parameters of a target resolution radar system.

SECTION VII

THE SPECIAL CASES

The special cases of the one- and two-dimensional inverse scattering problem (i.e., scattering data restricted to a κ-space line or plane respectively, obtained from a three-dimensional scatterer) can be treated by applying the methods of Lewis [12] or of this author [13] to (30) et seq. Namely, by choosing the characteristic aperture function for the two- and one-dimensional special cases respectively as

$$C(\kappa) = C(\kappa_1,\kappa_2)\ \delta(\kappa_3) \tag{61}$$

$$C(\kappa) = C(\kappa_1)\ \delta(\kappa_2)\ \delta(\kappa_3) \tag{62}$$

The three-dimensional convolution integral equation (38) reduces respectively to the two- and one-dimensional integral equations

$$a(x_1,x_2)*\beta(x_1,x_2) = g(x_1,x_2) \tag{63}$$

$$a(x_1)*\alpha(x_1) = g(x_1) \tag{64}$$

where β and α are the *thickness distribution function* in the x_3-direction and the *area distribution function* orthogonal to the x_1-direction of the scatterer respectively, and $g(x_1,x_2)$ and $g(x_1)$ reduce respectively to (see (37) et seq.)

$$g(x_1,x_2) = \frac{1}{(2\pi)^2}\int_C e^{i(\kappa_1 x_1+\kappa_2 x_2)}\ \Gamma(\kappa_1,\kappa_2)\ W(\kappa_1,\kappa_2)\ d\kappa_1\ d\kappa_2 \tag{65}$$

$$g(x_1) = \frac{1}{2\pi}\int_C e^{i\kappa_1 x_1}\,\Gamma(\kappa_1)\,W(\kappa_1)\,d\kappa_1 \tag{66}$$

The non-singular integral equations (63) and (64) can be solved for β and α by any of the previously discussed means.

The further special case of a priori knowledge of the scatterer being a surface of revolution about the x_2-axis (see fig. 2) can clearly be treated by the preceding two-dimensional formulation (63) and (65) by recognizing that the *profile function* (generatrix of revolution) of the scatterer is $\frac{1}{2}\beta(0,x_2)$; thereby further simplifying (63) and (65) after the appropriate modifications. A more direct treatment of the problem of the surface of revolution will be presented subsequently.

The special cases of the aperture $A(\kappa)$ being of certain given geometrical shapes can be treated by applying the method of Lewis [14] to (30) et seq.

The special cases of a priori knowledge of the scatterer possessing certain geometrical symmetries can be treated by applying the methods of this author [15] to (30) et seq. As an illustrative example, the case of the scatterer known to be a surface of revolution about the x_2-axis (see fig. 2), is presented next.

For such a scatterer, the monostatic cross-section clearly is independent of the longitudinal aspect angle ζ; i.e.

$$\rho(\kappa,\xi,\zeta) = \rho(\kappa,\xi)\ . \tag{67}$$

Furthermore, the profile function $x_3(x_2)$ of such a surface of revolution (see fig. 2) is given by the function describing the upper surface $Z(x_1,x_2)$ at the plane x_1=o. It thus follows from taking the limits of integration over ζ in (50) as from 0 to 2π, setting x_1=o in (50), and, with the aid of the integral representation of the Bessel functions, that

$$\chi(x_1,x_3) = \frac{1}{2\pi^{3/2}} \int_{\kappa_1}^{\kappa_2} \int_{-\xi_o}^{\xi_o} e^{i\kappa x_1 \sin\xi} J_1(\kappa x_3 \cos\xi)\, \rho(\kappa,\xi) \cos^2\xi \, d\xi \, \kappa \, d\kappa \, , \tag{68}$$

which for small angle approximations yields the doubly truncated two-dimensional mixed Hankel (Fourier-Bessel)-Fourier transform

$$\chi(x_1,x_3) = \frac{1}{2\pi^{3/2}} \int_{\kappa_1}^{\kappa_2} J_1(\kappa x_3) \int_{-\xi_o}^{\xi_o} e^{i\kappa\xi x_1} \rho(\kappa,\xi) \, d\xi \, \kappa \, d\kappa \, . \tag{69}$$

It is noteworthy that, as expected intuitively, only two-dimensional scattering information (in κ and ξ; i.e., in frequency and one aspect angle, the latitudinal aspect angle) is required for an inverse scattering solution by (68) or (69).

Solutions are also obtainable for the various combinations of the special cases cited.

SECTION VIII

NUMERICAL RESULTS

The solution to the integral equation (38) proposed by Lewis [11] was successfully numerically tested for a sphere by this author in 1969. This test consisted of a computer implementation of a special case version of this solution, applicable only to scatterers about which only a priori knowledge of cylindrical symmetry exists. This test essentially confirms the correctness of the basic inverse scattering identity (25) and the finite aperture integral equation (38). This solution was, however, not pursued further because of its inherent practical limitations. These limitations are the lack of generality of the required k-space aperture (i.e., the required aperture is impractical for physically realizable radar systems; which is not the case with this author's solution 50), the error enhancement introduced by the process of numerical differentiation of noisy scattering data (vis-a-vis the error reduction resulting from the process of integration of such data in solution 50), and the unapplicability of the Fast Fourier Transform (FFT) to this solution (which is essential if large amounts of data are to be processed in reasonable time by existing computers, yielding three-dimensional high-resolution descriptions of arbitrarily shaped scatterers about which no a priori knowledge of special geometry exists).

Solution (50) was computer implemented with the aid of the FFT for arbitrarily shaped apertures, realizable with existing radar systems. This computer program was tested with the exact solution of Mie for scattering by a sphere, with a variety of band limited aspect angles and fractional frequency band widths β (see fig. 3), with the results shown in fig. 5 through 8. The basic inverse scattering identity was also tested by this program; with results shown in fig. 4.

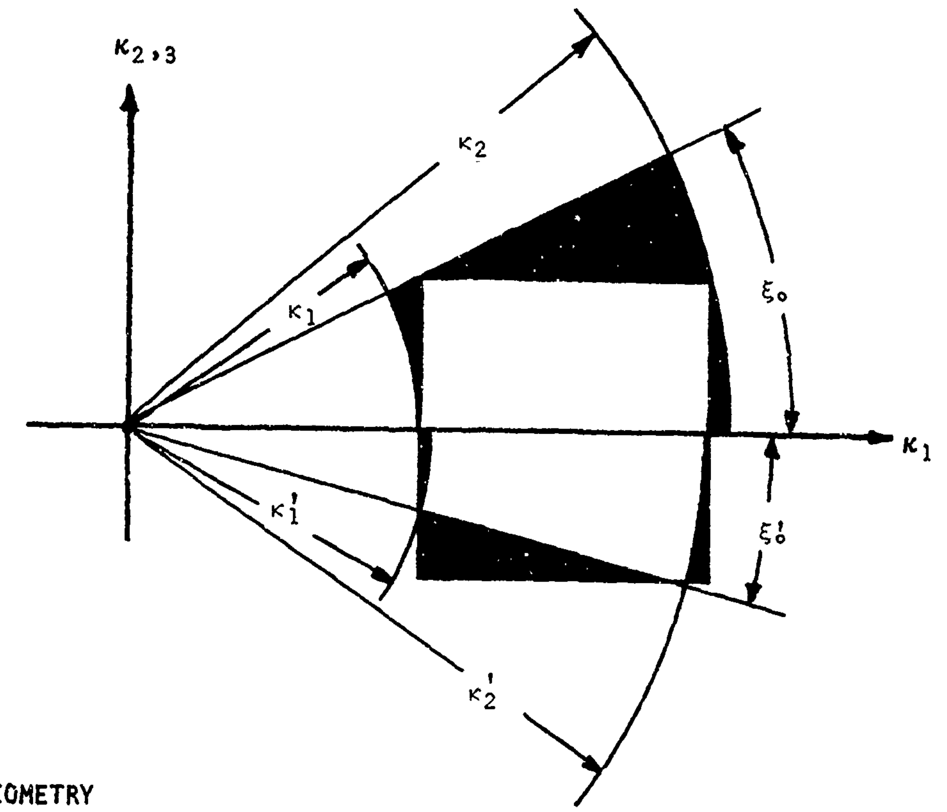

FIGURE 3

APERTURE GEOMETRY

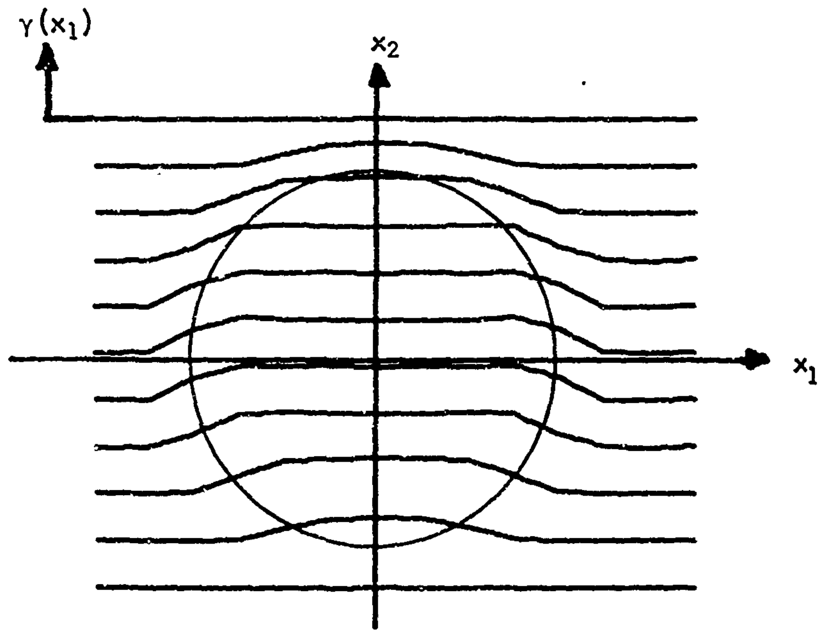

FIGURE 4

CHARACTERISTIC FUNCTION γ AT

β = 100 %

ζ_o = 180°

$x_3 = 0$

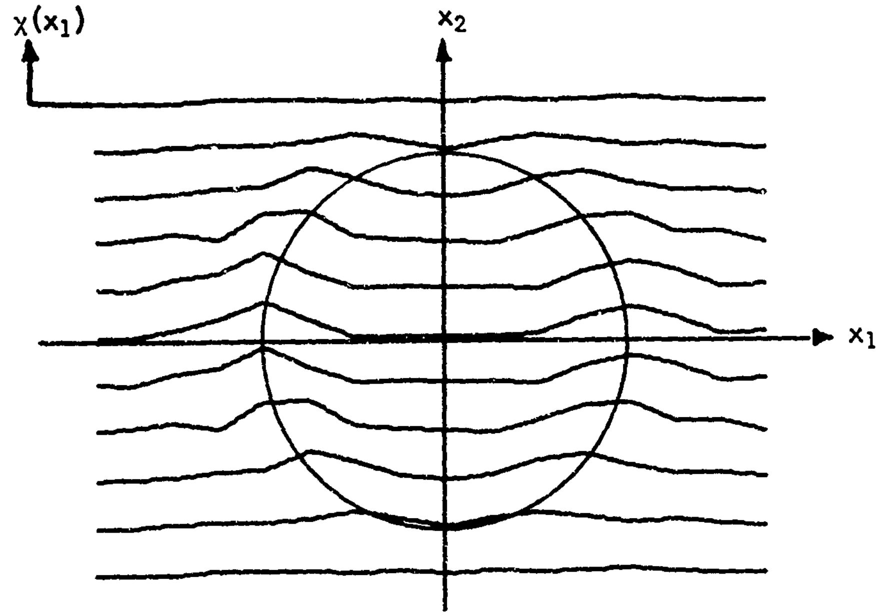

FIGURE 5

RESOLUTION DENSITY FUNCTION χ AT

β = 100 %
ξ_o = 180°
x_3 = o

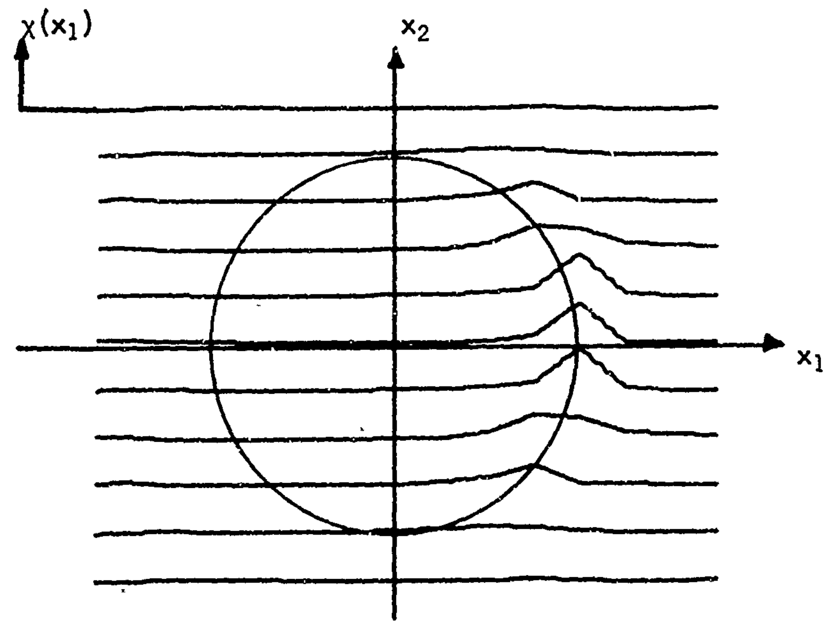

FIGURE 6

RESOLUTION DENSITY FUNCTION χ AT

β = 100 %
ξ_o = 90°
x_3 = o

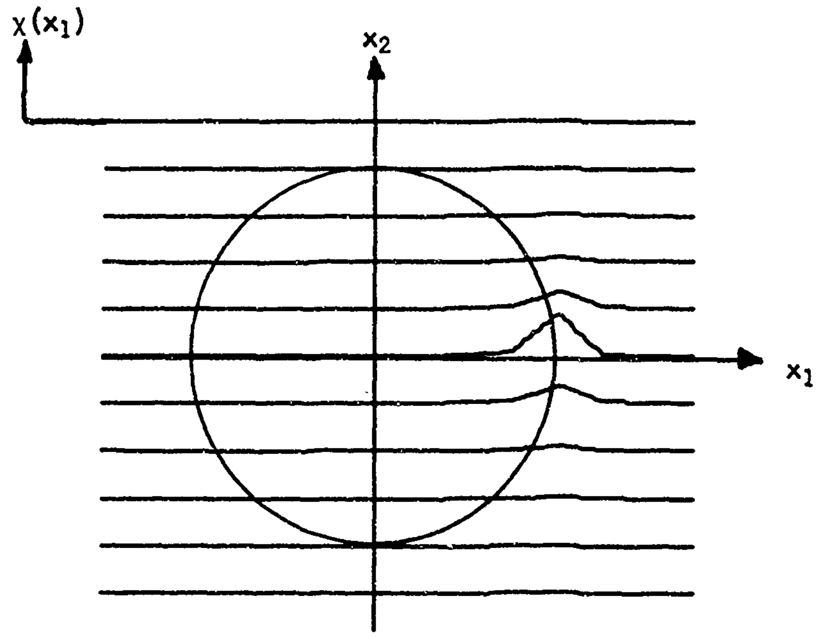

FIGURE 7

RESOLUTION DENSITY FUNCTION χ AT

$\beta' = 48.36$ %
$\xi_0' = 14.48°$
$x_3 = 0$

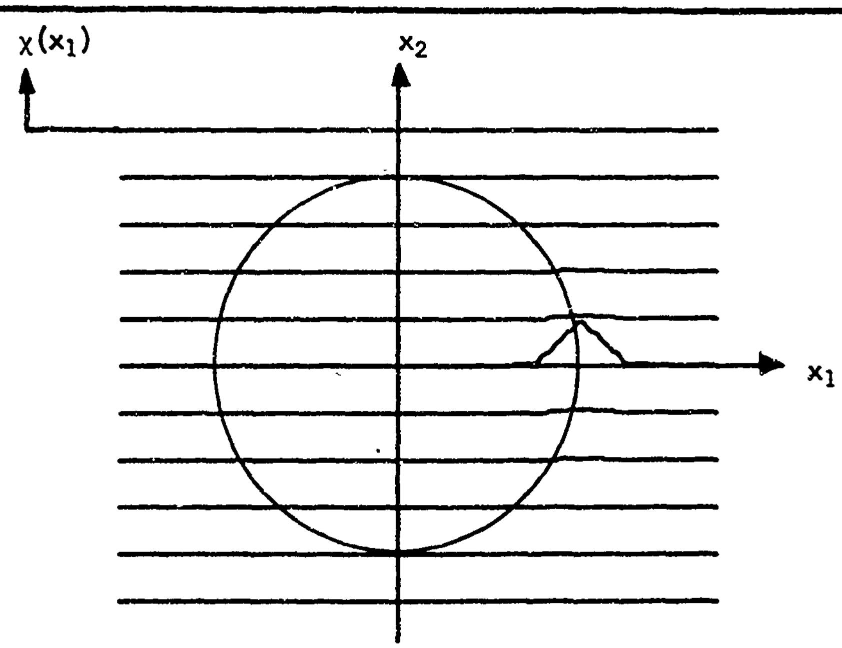

FIGURE 8

RESOLUTION DENSITY FUNCTION χ AT

$\beta' = 9.00$ %
$\xi_0' = 2.61°$
$x_3 = 0$

Only $x_3=0$ "slices" of the reconstructed scatterer are shown in fig. 4 through 8; however, several other than $x_3=0$ "slices" were also numerically reconstructed correctly, thus confirming the three-dimensional capabilities of solution (50). A full band-width three-dimensional display of a reconstructed sphere and cylinder are shown in fig. 9 and 10 respectively.

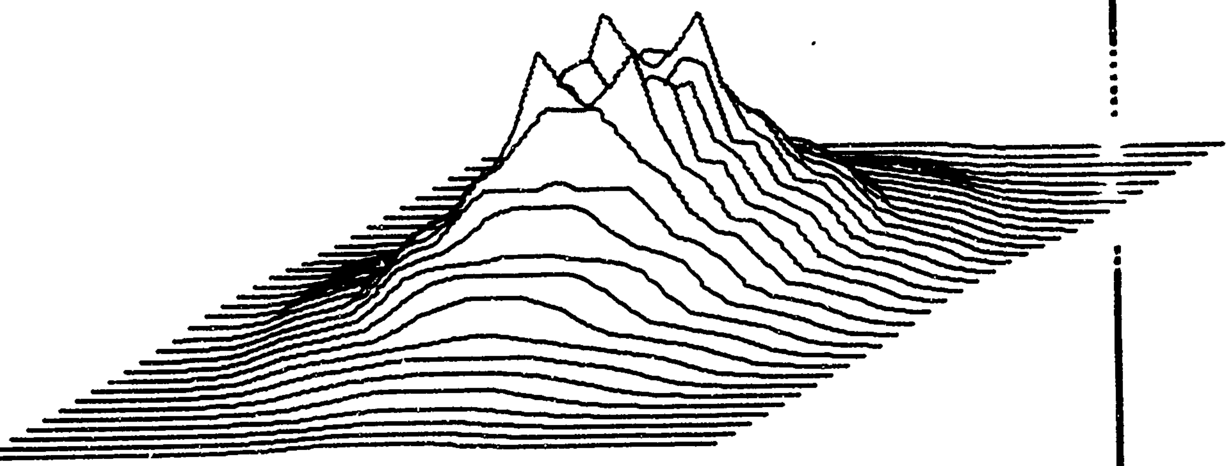

FIGURE 9
THREE-DIMENSIONAL DISPLAY OF RECONSTRUCTED SPHERE

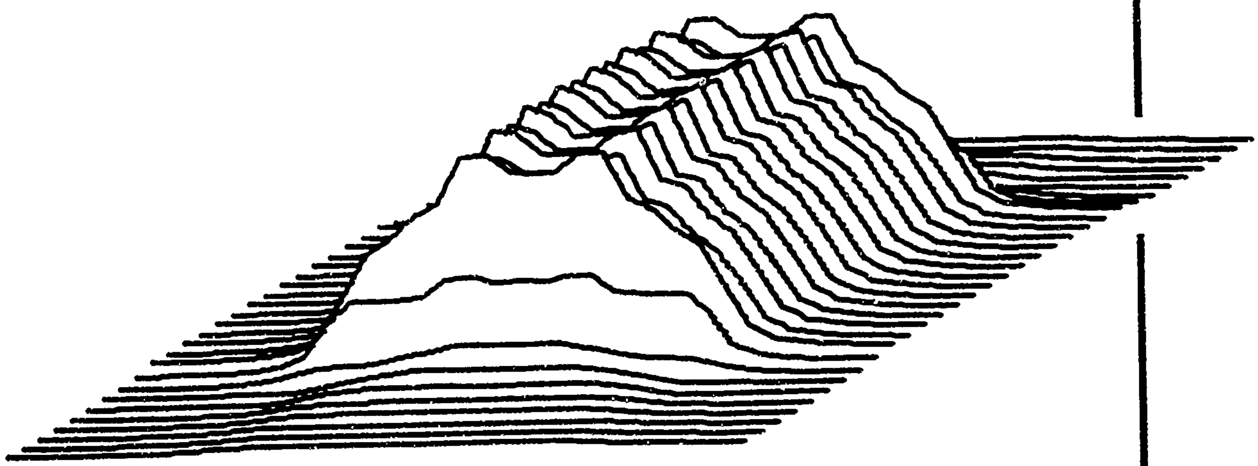

FIGURE 10
THREE-DIMENSIONAL DISPLAY OF RECONSTRUCTED CYLINDER

SECTION IX

PRESENT STATUS AND FUTURE WORK PLANS

The fast Fourier transform computer program utilized in the numerical work summarized in sect. 8 was an in-core FFT program limited to about 16,000 data points. For increased resolution, particularly at high frequencies, low band width, and small angular apertures, it is necessary to have an FFT program operative on about $(100)^3 = 10^6$ data points. Since data of this size cannot be stored in core, and must be stored externally (e.g., drum or disc), an external-storage-FFT program is currently being implemented under a follow-on contract. Under this follow-on contract, the inverse scattering solution utilizing such an external-storage FFT will also be tested against actual measured scattering data from a variety of modeled shapes of varying complexity.

The physical optics inverse scattering solution developed in this part I has been extended from the presented monostatic solution for stationary perfectly conducting scatterers to a generalized monostatic-bistatic inverse scattering solution for moving scatterers of arbitrary (unknown) conductivity. This generalized physical optics solution turns out to be identical with the first order solution of the exact inverse scattering solution presented in part II. It is for this reason that this generalization is not presented here.

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SECTION X

CONCLUSIONS

The solutions developed in this part I can be viewed as solutions to the frequency band limited and aspect angles limited short pulse synthetic aperture radar imaging (and associated data processing) problem. The solutions presented are based on rigorous electromagnetic scattering and inverse scattering theory applicable to spatially distributed scatterers, yielding real three-dimensional geometrical images; vis-a-vis the conventional two-dimensional radar imaging technique which is based on the heuristic approach of isolated point scatterers (scattering centers) concepts, yielding so-called radar images (or maps) [16], which most often do not resemble the real geometrical images of the scatterer.

The merits of the solutions presented in this paper over the conventional radar imaging technique deserve the following further examination. The rather unsatisfactory results of the conventional radar imaging technique are essentially the consequence of the technique being based on a heuristic approach to the problem; i.e., the approach consists of considering a spatially extended target as a fictitious ensemble of identifiable, stationary, non-interactive, non-dispersive, and isotropic point scatterers. From a rigorous electromagnetic scattering point of view, a spatially extended scatterer is neither an ensemble of point scatterers, nor are these fictitious point scatterers in principle identifiable, stationary, non-interactive, non-dispersive, and/or isotropic scatterers. (The point scatterers are not always identifiable by virtue of the so-called registration problem; i.e., that the point scatterers can be continuously, consistently, and correctly identified for various aspect angles. The point scatterers are not always stationary due to the fictitious relocation caused by changing aspect angles.) Furthermore, this technique does not contain basic unique existence considerations, and/or rigorous considerations of the problem of optimizing the results for incomplete aspect angles and/or frequency information availability. The attempts to convert radar images to geometrical images have thus failed for precisely these reasons.

The solutions of this part I alleviate all these objections to the so-called radar imaging technique by the rigorous application of electromagnetic inverse scattering theory, based on rigorous direct scattering theory (and not based on the heuristic model of a spatially extended scatterer as a fictitious ensemble of identifiable, stationary, non-interactive, non-dispersive, isotropic point scatterers). It, therefore, avoids the problem of the conversion of radar images to geometrical images, by sidestepping and avoiding the radar image altogether, and addressing itself to the problem of generating actual three-dimensional geometrical images directly from radar data; including unique optimal results from incomplete observation aspect angles and frequency information.

PART II

EXACT INVERSE SCATTERING

SECTION I

INTRODUCTION

In section II, the general inverse scattering and radiation problem associated with the three-dimensional inhomogeneous scalar field Helmholtz wave equation is formulated as a Fredholm integro-differential equation of the second kind for the unknown fields and sources in the interior of a closed surface in terms of the known fields on this closed surface. It is shown in section III that the known ansatz to this integro-differential equation reduces to an angular spectrum Fourier integral if the closed surface integral is taken over the far fields at infinity. In section IV, this far field integro-differential equation is reformulated as a purely algebraic equation in the spatial Fourier-transform k-space, yielding a purely algebraic closed form solution for the unknown fields in this k-space. This solution is then transformed back into the spatial domain, yielding a closed-form solution to the far-field integro-differential equation, consisting of a single integral resolvent operator, the resolvent kernel of which is the steady state Green's function of the Poisson equation associated with the inhomogeneous Helmholtz equation. This scalar inverse theory is clearly applicable to quantum mechanics, acoustics, etc.

In sect. V, the inverse integro-differential equation and its solution are generalized to the vector wave equation resulting from Maxwell's equations.

A formal physical and mathematical definition of the inverse problem is given in sect. VI; which, in essence, defines the inverse problem as that of unknown boundary conditions and/or constitutive equations which define the specific problem, given the general laws and the specific solutions which are the observable phenomenology.

The uniqueness, completeness, and well-behavedness of the inverse solution is discussed briefly in sect. VII.

It is shown in sect. VIII that the first order Neumann series solution of the inverse integro-differential equation, as well as the first order approximation of the exact solution of this inverse integro-differential equation, is identical with the equations governing synthetic microwave holography. The physical interpretation of this first order approximation is discussed, and shown to be equivalent to the physical optics approximation.

A fast Fourier transform method for evaluating numerically the exact solution to the inverse problem is presented in sect. IX.

In sect. X, the three-dimensional inverse integro-differential and its closed form solution are generalized to four dimensions, with the surprisingly simple result of the resolvent operator being algebraic. This four-dimensional solution is clearly applicable to Doppler-contaminated fields scattered or radiated by moving scatterers.

Since a distribution theory approach was taken in sect. II through X, appendix I consists of a simplified and unified rederivation of the classical Kirchhoff solution to the direct radiation problem, presented for the sake of convincing the reader of the validity, elegance, and power of this distribution theory approach.

Appendix II consists of the reformulation of Maxwell's equations into a form containing only the fundamental electric and magnetic fields (vis-a-vis the form also containing the displacement and induction fields) and the total charge and current densities (vis-a-vis the form containing the free charge and current densities only), thus yielding an electromagnetic wave equation universally valid for all media. Also developed in this appendix II is a suitably generalized constitutive equation - boundary condition relating this total current density to those fundamental electromagnetic fields. Both these reformulations are essential for the subject matter covered in sect. V, et. seq.

SECTION II

THE INVERSE SCATTERING INTEGRAL EQUATION

Gauss' theorem, when applied to the fector field $\nabla(g\phi)$, yields

$$\oint_s d\mathbf{S}\cdot\nabla(g\phi) = \int_v dv\ \nabla\cdot\nabla(g\phi) \tag{1}$$

$$= \int_v dv\ \nabla^2(g\phi)\ , \tag{2}$$

provided $\nabla(g\phi)$ is continuous and, hence, differentiable in v and on s. These continuity conditions can be totally dispensed with if the scalar fields ϕ and g, and, hence, $\nabla(g\phi)$, are taken as distributions [17]; specifically, if ϕ is taken as a field satisfying the inhomogeneous (neither source- nor singularity-free) Helmholtz (time-reduced wave) equation

$$\nabla^2\phi + k_0^2\ \phi = -\rho\ , \tag{3}$$

and g is taken as the associated free-space Green's function

$$g(\mathbf{x}|\mathbf{x}') = \frac{e^{ik_0 r}}{4\pi r}\ , \tag{4}$$

$$\mathbf{r} \equiv \mathbf{x}-\mathbf{x}'\ , \tag{5}$$

$$r \equiv |\mathbf{r}|\ , \tag{6}$$

which satisfies the inhomogeneous differential equation

$$\nabla^2 g + k_0^2\, g = -\,\delta\ , \tag{7}$$

and where both ϕ and g satisfy the radiation condition at infinity. (The free-space wave number $\frac{\lambda}{2\pi} = \frac{\omega}{c}$ is designated by k_0, vis-a-vis the conventional notation k, since the latter will subsequently be used to designate the Fourier transform variable of the spatial coordinate x).

In order to further convince the reader not familiar with distribution theory and its validity and power when applied to field problems, this same distribution theory approach is applied in appendix I to the rederivation of the classical Kirchhoff method of integration of the field equations for the direct radiation (and scattering) problem, without imposing the continuity restrictions, and, hence, without the classical singularity isolation spherical surface which is taken to the limit of vanishing size. The derivation of the inverse scattering integral equation presented in this sect. II can also be accomplished by such classical means; however, the resulting mathematical derivation is beset with vastly increased cumbersome details, which only obscure the physical meanings involved.

The left-hand side of (1) yields

$$\oint_S d\mathbf{S}\cdot\nabla(g\phi) = \oint_S (g\ \nabla\phi + \phi\ \nabla g)\cdot d\mathbf{S} \tag{8}$$

$$= \oint_S (g\ \frac{\partial\phi}{\partial n} + \phi\ \frac{\partial g}{\partial n})\, ds\ , \tag{9}$$

and the right-hand side of (1) yields

$$\int_V dv\ \nabla\cdot\nabla(g\phi) = \int_V dv\ \nabla\cdot(g\ \nabla\phi + \phi\ \nabla g) \tag{10}$$

$$= \int_V dv\ [\nabla\cdot(g\ \nabla\phi) + \nabla\cdot(\phi\ \nabla g)] \tag{11}$$

$$= \int_V dv\ (\nabla g\cdot\nabla\phi + g\ \nabla^2\phi + \nabla\phi\cdot\nabla g + \phi\ \nabla^2 g) \tag{12}$$

$$= \int_V dv\ (g\ \nabla^2\phi + \phi\ \nabla^2 g + 2\nabla\phi\cdot\nabla g)\quad , \tag{13}$$

which, with the aid of (3) and (7) yields

$$\int_V dv\ \nabla^2(g\phi) = \int_V dv\ [g(-k_o^2\phi-\rho) + \phi(-k_o^2 g-\delta) + 2\nabla g\cdot\nabla\phi] \tag{14}$$

$$= \int_V dv\ (-k_o^2 g\phi - g\rho - k_o^2 g\phi - \delta\phi + 2\nabla g\cdot\nabla\phi) \tag{15}$$

$$= \int_V dv\ (2\nabla g\cdot\nabla\phi - 2k_o^2\ g\ \phi - g\ \rho - \delta\ \phi) \tag{16}$$

$$= \int_V dv(2\nabla g\cdot\nabla\phi - 2k_o^2 g\phi) - \int_V dv\ g\rho - \int_V dv\ \delta\phi\quad . \tag{17}$$

If the field-point X' is taken as inside the volume bound by the surface s, then

$$\int_V dv\ \delta\ \phi = \phi'\quad . \tag{18}$$

By the conventional Kirchhoff interpretation of (3) and (7) for the internal (to s) field point X' (see appendix I),

$$\int_V dv\ g\ \rho = \phi' + \oint_S (\phi\ \nabla g - g\ \nabla\phi)\cdot d\mathbf{S} \quad , \tag{19}$$

where the right-hand side Kirchhoff surface integral of (13) represents the contribution to the field at the field point $\mathbf{X}'$ inside s due to sources ρ outside of s.

Thus, if all the sources are taken as in v (i.e., inside s; i.e., $\rho(\mathbf{X})=0$ for all $\mathbf{X}$ outside s), then (19) yields

$$\int_V dv\ g\ \rho = \phi' \quad . \tag{20}$$

By a similar argument, if the sources outside of s do not vanish, then these sources external to s must be taken as giving rise to an externally imposed (external to the inverse problem) incident field ϕ_i', given by (19) as

$$\phi_i' = -\oint_S (\phi\ \nabla g - g\ \nabla\phi)\cdot d\mathbf{S} \quad . \tag{21}$$

Equation (19) thus yields

$$\int_V dv\ g\ \rho = \phi' - \phi_i' \tag{20.1}$$

$$= \phi_s' \quad , \tag{20.2}$$

where ϕ_s' is the scattered field at the field point $\mathbf{X}'$. Since the scattered field also satisfies the Helmholtz equation (3), equations (1) through (20) remain valid if the field ϕ is taken as the scattered field ϕ_s only. Thus no formal distinction arises between the inverse radiation (active) and inverse scattering (passive) problems.

With the aid of (18) and (20), (17) reduces to

$$\int_V dv\ \nabla^2(g\phi) = 2\int_V dv\ (\nabla g\cdot\nabla\phi - k_o^2\ g\ \phi) - 2\phi' \quad . \qquad (22)$$

With the aid of (8) and (22), (2) reduces to

$$\phi' - \int_V dv\ (\nabla g\cdot\nabla\phi - k_o^2\ g\ \phi) = -\tfrac{1}{2}\oint_S (g\ \nabla\phi + \phi\ \nabla g)\cdot d\mathbf{s} \quad . \qquad (23)$$

In the context of the inverse problem, the integrand of the right-hand side surface integral of (23) is measurable, and, hence, this surface integral is evaluable; it thus becomes convenient to take this right-hand side of (23) as the known ansatz θ to the inverse problem; i.e.,

$$\boxed{\theta' \equiv -\tfrac{1}{2}\oint_S (g\ \nabla\phi + \phi\ \nabla g)\cdot d\mathbf{s} \quad .} \qquad (24)$$

By (23) and (24), the inverse problem can thus be stated by the Fredholm integro-differential equation of the second kind

$$\boxed{\phi' - \int_V dv\ (\nabla g\cdot\nabla\phi - k_o^2\ g\ \phi) = \theta' \quad .} \qquad (25)$$

It should be noted that the classical Kirchhoff surface integral (see appendix I) vanishes in general if both the field point as well as all the sources are inside the surface (i.e., $\mathbf{X}'\in s$ and $\rho(\mathbf{X})=0$ for all $\mathbf{X}\notin s$), where as the ansatz surface integral (24) does not vanish in general for these same conditions. The physical reason for the vanishing of the Kirchhoff surface integral is clearly that there are no sources outside of this surface; since this surface integral represents the fields produced by sources outside of this surface. The mathematical reason for this vanishing is simply that the (-) sign in the integrand renders the two integral terms as canceling each other. Whereas the ansatz surface integral does not vanish since these same two terms add up by virtue of the (+) sign in the integrand (if the ansatz is in the far field, then these two integrands (vis-a-vis integrals) are equal). It is indeed remarkable that the sole distinction between these two surface integrals should turn out to be the (+) vs. (-) sign in the integrand, and that this sole transition from (-) to (+) should render the Kirchhoff surface integral as a useful ansatz (surface integral) to the inverse problem.

SECTION III

THE FAR-FIELD INVERSE SCATTERING INTEGRAL EQUATION

If only the far-field is known, then the surface integral of the ansatz (24) can clearly be taken as a spherical surface at infinity. Let R be the radius of this surface (see fig. 11); it thus follows that

$$\theta' = -\tfrac{1}{2} \underset{R \to \infty}{Limit} \int_{\Omega} (g\ \nabla\phi + \phi\ \nabla g)\cdot\hat{\mathbf{R}}\ R^2\ d\Omega \quad , \tag{26}$$

where the implied integration is now over the complete solid angle Ω.

The gradient of the Green's function (see eqn. (4)) is

$$\nabla g = \frac{e^{ik_\circ r}}{4\pi r^2} (ik_\circ r - 1)\ \hat{\mathbf{r}} \quad , \tag{27}$$

which, in the far-field (retaining only terms in $\frac{1}{r}$) reduces to

$$\nabla g = ik_\circ \hat{\mathbf{r}}\ \frac{e^{ik_\circ r}}{4\pi r} \tag{28}$$

$$= i\mathbf{k}_\circ\ \frac{e^{ik_\circ r}}{4\pi r} \quad , \tag{29}$$

where $\mathbf{k}_\circ$ is the far-field observation propagation wave-number vector (see fig. 11).

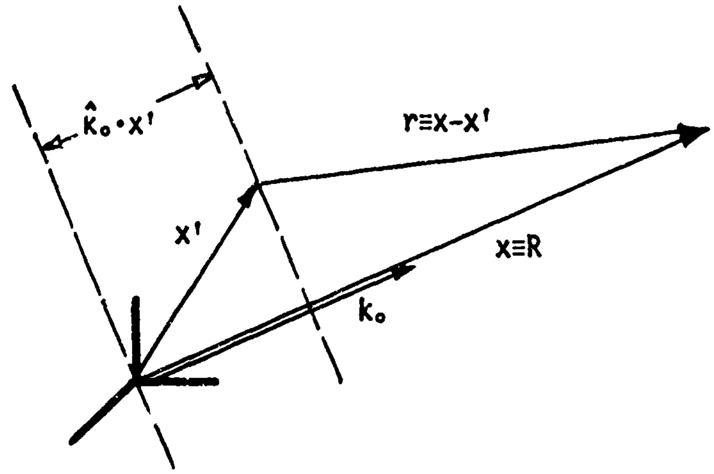

FIGURE 11
FAR-FIELD GEOMETRY

The field ϕ in the far-field (retaining only terms in $\frac{1}{R}$), can be expressed by virtue of the radiation condition at infinity, as (see fig. 11)

$$\boxed{\phi(x) = \frac{e^{ik_oR}}{\sqrt{4\pi}\,R}\ \psi(k_o)\quad ,} \tag{30}$$

where $\psi(k_o)$ is now the range and phase normalized (relative to a given coordinate system in which the far-field is measured and the near field ϕ is to be determined; see fig. 11). $\psi(k_o)$ clearly depends on the observation direction k_o only. The factor $\sqrt{4\pi}$ was chosen for the sake of consistency with the conventional definition of power cross section.

The gradient of this far-field (30) is thus

$$\nabla\phi = \frac{e^{ik_oR}}{\sqrt{4\pi}\,R^2}\ (ik_or - 1)\ \hat{R}\ \psi(k_o) + \frac{e^{ik_oR}}{\sqrt{4\pi}\,R}\ \nabla\psi(k_o)\quad , \tag{31}$$

which, in the far-field (retaining only terms in $\frac{1}{R}$), reduces to

$$\nabla\phi = \frac{e^{ik_o R}}{\sqrt{4\pi}\ R}\ [ik_o\hat{R}\ \psi(\mathbf{k}_o) + \nabla\psi(\mathbf{k}_o)] \tag{32}$$

$$= \frac{e^{ik_o R}}{\sqrt{4\pi}\ R}\ [i\mathbf{k}_o\psi(\mathbf{k}_o) + \nabla\psi(\mathbf{k}_o)] \quad . \tag{33}$$

With the aid of (4), (29), (30), and (33), (26) reduces to

$$\theta' = -\tfrac{1}{2} \mathop{Limit}_{R\to\infty} \int_\Omega \frac{e^{ik_o r}}{4\pi r} \frac{e^{ik_o R}}{\sqrt{4\pi}\ R} [2i\mathbf{k}_o\psi(\mathbf{k}_o)+\nabla\psi(\mathbf{k}_o)]\cdot\hat{R}\ R^2\ d\Omega \tag{34}$$

$$= -\tfrac{1}{2}\ (4\pi)^{-3/2} \mathop{Limit}_{R\to\infty} \int_\Omega e^{ik_o(r+R)}\ \frac{R}{r}\ [2i\mathbf{k}_o\cdot\hat{R}\ \psi(\mathbf{k}_o)+\hat{R}\cdot\nabla\psi(\mathbf{k}_o)]\ d\Omega \quad . \tag{35}$$

Since $\psi(\mathbf{k}_o)$ varies only in the angular (observation) direction, and not in the radial direction, it follows that (see fig. 11)

$$\hat{R}\cdot\nabla\psi(\mathbf{k}_o) = 0 \quad . \tag{36}$$

Also (see fig. 11)

$$\mathbf{k}_o\cdot\hat{R} = k_o \quad . \tag{37}$$

Equation (35) thus reduces to

$$\theta' = -ik_o\ (4\pi)^{-3/2} \mathop{Limit}_{R\to\infty} \int_\Omega e^{ik_o(r+R)}\ \frac{R}{r}\ \psi(\mathbf{k}_o)\ d\Omega \quad . \tag{38}$$

Examination of fig. 11 readily reveals that

$$\underset{R \to \infty}{Limit} \frac{R}{r} e^{ik_o(r+R)} = e^{2ik_oR} e^{-i\mathbf{k}_o \cdot \mathbf{X}'} \quad . \tag{39}$$

Equation (38) thus reduces to

$$\theta' = - ik_o \ (4\pi)^{-3/2} \ e^{2ik_or} \int_\Omega e^{-i\mathbf{k}_o \cdot \mathbf{X}'} \ \psi(\mathbf{k}_o) \ d\Omega \quad . \tag{40}$$

Since the arbitrary phase factor of ($2k_oR$) is common to all the measured ansatz, it becomes convenient to arbitrarily chose it as an integer multiple of 2π, thereby reducing (40) to

$$\theta' = - ik_o \ (4\pi)^{-3/2} \int_\Omega e^{-i\mathbf{k}_o \cdot \mathbf{X}'} \ \psi(\mathbf{k}_o) \ d\Omega \quad . \tag{41}$$

It further becomes convenient to now interchange the variables $\mathbf{X}$ and $\mathbf{X}'$ in (25) and (41); thus yielding the far-field integro-differential equation and its ansatz respectively as

$$\phi(\mathbf{X}) - \int [\nabla' g(\mathbf{X}|\mathbf{X}') \cdot \nabla' \phi(\mathbf{X}') - k_o^2 \ g(\mathbf{X}|\mathbf{X}') \ \phi(\mathbf{X}')] \ dv' = \theta(\mathbf{X}) \tag{42}$$

$$\theta(\mathbf{X}) = - ik_o \ (4\pi)^{-3/2} \int_\Omega e^{-i\mathbf{k}_o \cdot \mathbf{X}} \ \psi(\mathbf{k}_o) \ d\Omega \quad , \tag{43}$$

where the volume integral in (42), bound by the surface at infinity, is now over all space.

SECTION IV

SOLUTION OF THE FAR-FIELD INVERSE SCATTERING INTEGRAL EQUATION

In cartesian coordinates, the Green's function $g(X|X')$ can be written as (see (4), (5), and (6))

$$g(X|X') = g(X-X') \quad . \tag{44}$$

The integro-differential equation (42) can thus be written as the three-dimensional convolution equation

$$\phi(X) - \iiint_{-\infty}^{\infty} [\nabla' g(X-X') \cdot \nabla' \phi(X') - k_o^2 \, g(X-X') \, \phi(X')] \, d^3X' = \theta(X) \quad , \tag{45}$$

which, in a more compact notation, is

$$\phi - \nabla g \overset{*}{\cdot} \nabla\phi + k_o^2 \, g * \phi = \theta \quad , \tag{46}$$

where the operation $(\overset{*}{\cdot})$ designates the vector dot-product convolution.

If the fields and the Green's function in (46) are taken as distributions for the sake of the existence of their three-dimensional Fourier transform; i.e.,

$$\phi(\mathbf{X}) \leftrightarrow \Phi(\mathbf{k}) \tag{47}$$

$$\theta(\mathbf{X}) \leftrightarrow \Theta(\mathbf{k}) \tag{48}$$

$$g(\mathbf{X}) \leftrightarrow G(\mathbf{k}) \quad , \tag{49}$$

and, by the three-dimensional differentiation rule for Fourier transforms

$$\nabla\phi(\mathbf{X}) \leftrightarrow i\mathbf{k}\ \Phi(\mathbf{k}) \tag{50}$$

$$\nabla g(\mathbf{X}) \leftrightarrow i\mathbf{k}\ G(\mathbf{k}) \quad , \tag{51}$$

then, by the three-dimensional convolution theorem, the three-dimensional Fourier transform of (46) yields in the three-dimensional transform k-space the algebraic equation

$$\Phi(\mathbf{k}) + k^2\ G(\mathbf{k})\ \Phi(\mathbf{k}) + k_o^2\ G(\mathbf{k})\ \Phi(\mathbf{k}) = \Theta(\mathbf{k}) \tag{52}$$

$$[1 + (k^2 + k_o^2)\ G\ ]\ \Phi = \Theta \quad . \tag{53}$$

Furthermore, since for the three-dimensional delta-function taken as a distribution

$$\delta(\mathbf{X}) \leftrightarrow 1 \quad , \tag{54}$$

it follows that the three-dimensional Fourier transform of (7) yields in k-space the algebraic equation

$$- k^2 \, G(\mathbf{k}) + k_0^2 \, G(\mathbf{k}) = -1 \quad . \tag{55}$$

Thus

$$G(\mathbf{k}) = \frac{1}{k^2 - k_0^2} \quad . \tag{56}$$

With the aid of (56), (53) yields

$$\Phi(\mathbf{k}) + (k^2 + k_0^2) \, G(\mathbf{k}) \, \Phi(\mathbf{k}) = \Theta(\mathbf{k}) \tag{57}$$

$$\Phi(\mathbf{k}) + \left(\frac{k^2 + k_0^2}{k^2 - k_0^2} \right) \Phi(\mathbf{k}) = \Theta(\mathbf{k}) \tag{58}$$

$$\left(1 + \frac{k^2 + k_0^2}{k^2 - k_0^2} \right) \Phi(\mathbf{k}) = \Theta(\mathbf{k}) \tag{59}$$

$$\left(\frac{k^2 - k_0^2}{k^2 - k_0^2} + \frac{k^2 + k_0^2}{k^2 - k_0^2} \right) \Phi(\mathbf{k}) = \Theta(\mathbf{k}) \tag{60}$$

$$\left(\frac{2k^2}{k^2 - k_0^2} \right) \Phi(\mathbf{k}) = \Theta(\mathbf{k}) \tag{61}$$

$$\Phi(\mathbf{k}) = \left(\frac{k^2 - k_0^2}{2k^2} \right) \Theta(\mathbf{k}) \tag{62}$$

$$= \tfrac{1}{2} \, \Theta(\mathbf{k}) - \frac{k^2}{2k_0} \, \Theta(\mathbf{k}) \quad . \tag{63}$$

Next, the steady-state Green's function g_0, given by

$$g_o(\mathbf{X}) = \frac{1}{4\pi r} \quad , \tag{64}$$

satisfies the Poisson differential equation (associated with the Helmholtz equation (3))

$$\nabla^2 g_o(\mathbf{X}) = - \delta(\mathbf{X}) \quad . \tag{65}$$

Again, taking this steady-state Green's function as a distribution, assures the existence of the three-dimensional Fourier transform of this steady-state Green's function; i.e.,

$$g_o(\mathbf{X}) \leftrightarrow G_o(\mathbf{k}) \quad . \tag{66}$$

It thus follows from (65) by the three-dimensional differentiation rule that

$$- k^2 G_o(\mathbf{k}) = - 1 \tag{67}$$

$$G_o(\mathbf{k}) = \frac{1}{k^2} \quad . \tag{68}$$

Equation (63) can thus be written as

$$\Phi(\mathbf{k}) = \tfrac{1}{2}\, \Theta(\mathbf{k}) - \frac{k_o^2}{2}\, G_o(\mathbf{k})\, \Theta(\mathbf{k}) \quad , \tag{69}$$

and by the three-dimensional convolution theorem, it thus follows that (still in cartesian coordinates only)

$$\phi(X) = \tfrac{1}{2}\,\theta(X) - \frac{k_o^2}{2}\, g_o(X)*\theta(X) \tag{70}$$

$$= \tfrac{1}{2}\,\theta(X) - \frac{k_o^2}{2} \iiint_{-\infty}^{\infty} g_o(X-X')\,\theta(X')\,d\,x' \quad ; \tag{71}$$

and, consequently, in any coordinate system

$$\phi(X) = \tfrac{1}{2}\,\theta(X) - \frac{k_o^2}{2} \int g_o(X|X')\,\theta(X')\,dv' \quad . \tag{72}$$

It thus follows by (64) that the solution to the integro-differential equation (42) is

$$\boxed{\phi(X) = \tfrac{1}{2}\,\theta(X) - \frac{k_o^2}{8\pi} \int \frac{1}{r}\,\theta(X')\,dv' \quad .} \tag{73}$$

It is remarkable that the resolvent kernel [18] of the integro-differential equation (42) should turn out to be the steady-state Green's function; the physical implications of this fact, which are yet unclear, should be explored.

SECTION V

THE GENERALIZATION FOR THE ELECTROMAGNETIC VECTOR FIELDS

The generalization of the previously developed scalar field inverse scattering theory to a vector field inverse scattering theory applicable to the electromagnetic field can be accomplished analogously to the generalization of Stratton and Chu [19] and Franz [20] of the Kirchhoff method of integration of the scalar field equations to vector fields, applicable to direct scattering or radiation; i.e., by replacing (1) with Gauss' theorem of the form

$$\oint_S d\mathbf{S}\times\nabla\times(g\mathbf{E}) = \int_V dv\ \nabla\times\nabla\times(g\mathbf{E}) \tag{74}$$

$$\oint_S d\mathbf{S}\times\nabla\times(g\mathbf{H}) = \int_V dv\ \nabla\times\nabla\times(g\mathbf{H}) \quad , \tag{75}$$

where the electric and magnetic vector fields $\mathbf{E}$ and $\mathbf{H}$ satisfying the vector wave equations

$$\nabla\times\nabla\times\mathbf{E} - k_o^2\ \mathbf{E} = i\omega\mu_o\ \mathbf{J} \tag{76}$$

$$\nabla\times\nabla\times\mathbf{H} - k_o^2\ \mathbf{H} = \nabla\times\mathbf{J} \quad , \tag{77}$$

which replace the scalar wave equation (3) used previously.

From here on, the generalization proceeds in a fashion totally analogous to the previously presented development; however, with the following one important exception. The scalar theory developed consists of a single field, and, hence, a single statement of Gauss' theorem (1) and a single wave equation (3). The vector theory consists of two fields (the electric and magnetic fields), and, hence, a dual statement of Gauss' theorem (74), (75) and two wave equations (76), (77). This dual set of vector equations, is, however, coupled by a common source (the current density J in the wave equations (76), (77)), as well as Maxwell's equations, which are a set of first-order differential equations for which there is no scalar analog, and which are used to entwine the coupling.

Regretably, however, such a procedure, although executable, tends to bog down in massive, cumbersome, and inelegant details, which tend to obscure the physical meanings involved. An alternate, much simpler derivation, which yields the identical results, will thus be presented next.

Prior to proceeding with this derivation, however, a digressionary discussion of the form of Maxwell's equation most appropriate to the inverse problem is in order. For organizational reasons, however, this discussion is relegated to appendix II, to which the reader is referred.

The universally valid time-reduced vector wave equation for the magnetic vector field H, in terms of the total currents J (see appendix II) is

$$\nabla\times\nabla\times H - k_o^2 H = \nabla\times J \quad , \tag{78}$$

which reduces to

$$\nabla\nabla\cdot H - \nabla^2 H - k_o^2 H = \nabla\times J \quad , \tag{79}$$

which, by Maxwell's second equation (see appendix II, eqn. (33)), further reduces to

$$\nabla^2 \mathbf{H} + k_0^2\, \mathbf{H} = - \nabla\times\mathbf{J} \quad . \tag{80}$$

If a source vector $\boldsymbol{\rho}$ is defined as

$$\boldsymbol{\rho} \equiv \nabla\times\mathbf{J} \quad , \tag{81}$$

then (80) can be written as

$$\nabla^2 \mathbf{H} + k_0^2\, \mathbf{H} = - \boldsymbol{\rho} \quad . \tag{82}$$

If H_μ and ρ_μ designate the cartesian components of the vectors $\mathbf{H}$ and $\boldsymbol{\rho}$ respectively (where μ=1,2,3.), then (82) can be rewritten in a mixed vector - cartesian tensor notation as

$$\nabla^2 H_\mu + k_0^2\, H_\mu = - \rho_\mu \quad ; \mu=1,2,3. \tag{83}$$

Each of the three equations (83) clearly is of the form of the scalar field wave equation (3), it thus follows that the entire previously developed scalar field inverse scattering theory is applicable to each of the three cartesian components of $\mathbf{H}$ and $\boldsymbol{\rho}$. The inverse scattering integro-differential equation (25) and its ansatz (24) can thus be written (still in mixed vector - cartesian tensor notation) respectively as

$$H'_\mu - \int_V dv\ (\nabla g \cdot \nabla H_\mu - k_o^2\ g\ H_\mu) = \theta'_\mu \qquad , \mu=1,2,3. \tag{84}$$

$$\theta'_\mu = -\ \tfrac{1}{2} \oint_S d\mathbf{s} \cdot (g\ \nabla H_\mu + H_\mu\ \nabla g) \qquad , \mu=1,2,3. \tag{85}$$

which clearly can be reconciled into the pure vector notation form

$$\boxed{\mathbf{H}' - \int_V dv\ (\nabla g \cdot \nabla \mathbf{H} - k_o^2\ g\ \mathbf{H}) = \boldsymbol{\theta}'} \tag{86}$$

$$\boxed{\boldsymbol{\theta}' = -\ \tfrac{1}{2} \oint_S d\mathbf{s} \cdot (g\ \nabla \mathbf{H} + \nabla g\ \mathbf{H}) \quad .} \tag{87}$$

Similarly, the far-field inverse scattering integral equation (42) and its ansatz (43) yield respectively

$$\boxed{\mathbf{H} - \int dv'\ (\nabla' g' \cdot \nabla' \mathbf{H}' - k_o^2\ g'\ \mathbf{H}') = \boldsymbol{\theta}} \tag{88}$$

$$\boxed{\boldsymbol{\theta} = -\ ik_o\ (4\pi)^{-3/2} \int_\Omega e^{-i\mathbf{k}_o \cdot \mathbf{x}}\ \boldsymbol{\psi}(\mathbf{k}_o)\ d\Omega \quad ,} \tag{89}$$

where, by (30), the range and phase normalized magnetic far-field $\boldsymbol{\psi}(\mathbf{k}_o)$ is given by

$$\boxed{\mathbf{H}(\mathbf{x}) = \frac{e^{ik_o R}}{\sqrt{4\pi}\ R}\ \boldsymbol{\psi}(\mathbf{k}_o) \quad .} \tag{90}$$

and again similarly, and finally, the solution (73) to this far-field inverse scattering integral equation yields

$$\boxed{H(x) = \frac{1}{2}\,\theta(x) - \frac{k_o^2}{8\pi}\int \frac{1}{r}\,\theta(x')\;dv' \quad .} \tag{91}$$

Unfortunately, however, no such simple derivation for the electric field has been found to date. However, commencing with (74), (76), et seq., a set of electric field equations totally analogous to (86) through (91) is derivable; i.e., a set of equations (86) through (91) in which the magnetic field H is replaced with the electric field E. The details of this derivation are not presented here because of their cumbersomeness, and because these details shed no light on the physical meanings involved.

SECTION VI

FORMAL STATEMENT AND SOLUTION OF THE INVERSE PROBLEM

An academic digression to a formal statement and definition of the *inverse problem* is undertaken next. First, however, a careful examination, formal restatement, and definition of the classical *direct* problem is in order.

From a mathematical point of view, a direct problem consists of a given differential equation, a given set of boundary conditions, and an unknown solution (or spectrum of solutions). From a physical point of view, the differential equation represents the universally valid law, the boundary conditions represent the nature of the specific problem, and the unknowns are the observable phenomenology. Additional insight is gained by examining the structure of the equivalent integral equation formulation of the direct problem. Mathematically, such integral equation formulations are in general arrived at in the following fashion. An integral representation of the differential equation is obtained (usually by an integration of this differential equation with the aid of an appropriate Green's function), which is then combined with the appropriate boundary conditions to yield an integral equation. Physically, the integral representation of the differential equation is merely an integral restatement of the universally valid law, which, in general, clearly illustrates its global nature (i.e., that fields at one point are determined by sources at all other points). Conversely, however, the boundary conditions are a restatement of the specific constitutive equations, which determine the specific problem. Such constitutive equations are in general local in nature (i.e., the specific relation between the sources at one point and the fields at that same point only). The following equivalence between boundary conditions and constitutive equations is thus

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of importance. Mathematically, a constitutive equation is a spatially distributed boundary condition; and, conversely, physically, a boundary condition is a localized constitutive equation. (Such boundary conditions - constitutive equations in general arise from a localized imposition of the appropriate laws of motion governing the source distribution.) From an integral equation point of view, it is this boundary condition - constitutive equation that determines and imposes the given known domain of integration, and, hence, the nature of the specific problem.

From a mathematical point of view, an inverse problem thus consists of a given differential equation or its equivalent integral representation, a given solution (or spectrum of solutions), and an unknown boundary condition (or its equivalent domain of integration). From a physical point of view, an inverse problem thus consists of a given universal law, a given (observed, measured) phenomenology, and an unknown constitutive equation.

Having defined the direct and inverse problems in the preceding fashion, it follows naturally that if the differential equation (or its integral representation) stating the universal law is the unknown, then the problem can be classified as the *basic* or *fundamental problem* (there exists no formal methodology to date for solving such problems).

The following short table is thus in order:

PROBLEM CLASSIFICATION

MATHEMATICAL ENTITY	PHYSICAL MEANING	IF UNKNOWN, PROBLEM IS
Differential Equation (or equivalent integral equation representation)	Universal (Global) Law	Basic, Fundamental
Boundary Condition (or domain of integration)	Constitutive (Local) Equation	Inverse
Specific Solution to differential (or integral) equation	Observable (Measurable) Phenomenology	Direct

The inverse scattering (or radiation) problem is thus the problem of determining the unknown boundary conditions - constitutive equations (i.e., the *structure* of the scatterer or radiator), given the field equations and the incident and scattered (or total radiated) fields. The problem of determining the unknown source distribution that gave rise to the observed scattered field (or radiation), although intimately connected with the inverse scattering (or radiation) problem, is thus only incidental to, and not the final objective of, the inverse scattering (or radiation) problem. The distinction between the inverse scattering and inverse radiation problem is the mathematical and physical triviality of whether the unknown structure is radiating passively or actively, respectively (i.e., whether the observed fields are measured actively or passively, respectively).

For the direct vs. the inverse scattering (and radiation) problem, the following specific additional mathematically fundamental topological distinction arises. For the direct problem, the Kirchhoff closed surface integral $\oint d\mathbf{S}\cdot(g\,\nabla\phi - \phi\,\nabla g)$ relates fields at a field point located on one side of the closed surface to all the source distribution located on the *other* side of this closed surface (see appendix I). For precisely this reason, since the inverse problem is characterized by unknown fields as well as unknown sources, this Kirchhoff surface integral is thus rendered totally useless to the inverse problem. However, the closed surface integral $\oint d\mathbf{S}\cdot(g\,\nabla\phi + \phi\,\nabla g)$ relates the fields at a field point located on one side of the closed surface to the source distribution located on the *same* side of this closed surface (see (24) and (25)). It is for this reason that this latter closed surface integral is rendered a useful ansatz to the inverse problem.

In the context of this definition of the inverse problem, it is thus clear that neither a solution for the electric and magnetic fields (see (86) through (91)), nor a solution for the total current distribution J that gave rise to these fields, is a solution to the inverse problem; however, a solution for the unknown boundary conditions - constitutive equations which locally connect these electric and magnetic fields to the current distribution, is a solution to the inverse problem. It is to this latter problem that the remainder of this section is addressed.

Given a solution for $\mathbf{E}$ and $\mathbf{H}$ by (86) through (91), the total current distribution $\mathbf{J}$ can be computed on a local basis via the universally valid Maxwell's fourth equation (see appendix II, eqn. (35) and (46)) for the total current; i.e.,

$$\mathbf{J} = - i\omega\varepsilon_o \mathbf{E} + \nabla\times\mathbf{H} \quad . \tag{92}$$

Thus, given the electric and magnetic fields, as well as the total current density, the coefficients of the generalized constitutive equation (see appendix II, eqn. (49)), which determine (reveal) the structure, geometry, and electromagnetic properties of the scatterer (or radiator), can be readily computed. In fact, since these scalar constitutive coefficients in the vector constitutive equation are over-determined by knowledge of the vector fields, a least-square-best-estimate (in the presence of noise) of these coefficients can be determined as well.

SECTION VII

UNIQUENESS, COMPLETENESS, AND WELLBEHAVEDNESS OF THE INVERSE SOLUTION

The problem of uniqueness, completeness, and wellbehavedness of the inverse solution (under incomplete and noisy error contaminated ansatz) is intimately connected with the algebraic nature of the k-space transform representations (57) and (69) of the inverse integral equation and its solution reprectively, the Fourier surface integral transform nature of the associated ansatz (43), and the extension of the entire inverse theory and its solution to the wide-band (short-pulse) case. For these reasons, both the method of derivation as well as the final results governing the uniqueness, completeness, and wellbehavedness of the solution are identical to the method of derivation and results governing the physical optics inverse scattering theory (see part I). A further detailed investigation and study of this subject is, however, currently being undertaken under a follow-on contract (to contract covered by this report). A detailed treatment of this subject is thus being deferred to subsequent report(s) covering this follow-on contract.

SECTION VIII

THE FIRST-ORDER APPROXIMATION

The first-order approximation Neumann series solution of the general inverse integro-differential equation (23), with the aid of (24), clearly is

$$\phi' = -\tfrac{1}{2}\oint_S (g\ \nabla\phi + \phi\ \nabla g)\cdot d\mathbf{S} \quad , \tag{93}$$

which, for the far-field ansatz case, with the aid of (43), reduces to

$$\phi(\mathbf{X}) = -\ ik_\circ\ (4\pi)^{-3/2} \int_\Omega e^{-i\mathbf{k}_\circ\cdot\mathbf{X}}\ \psi(\mathbf{k}_\circ)\ d\Omega \quad ; \tag{94}$$

which turns out to be identical to the first-order approximation of the exact far-field solution (73).

This first-order solution is identical to the *synthetic microwave holography* solution of Tricoles [21], which has already enjoyed considerable experimental verification. The derivation of this first-order solution can thus be taken as the formal theoretical foundation of this synthetic microwave holography.

In the derivation of the synthetic microwave holography equations, the "ansatz" is taken as $\oint d\mathbf{S}\cdot(g\ \nabla\phi^* - \phi^*\ \nabla g)$ where ϕ^* denotes the complex conjugate of the outwardly radiated field; which, in turn, heuristically represents the "direction of propagation reversed" equivalent "inwardly focussed" field. From a holographic point of view, this ansatz also could have been taken as $\oint d\mathbf{S}\cdot(g^*\ \nabla\phi - \phi\ \nabla g^*)$, where g^* denotes the complex conjugate of the "outgoing" Green's function; which in turn represents the "ingoing" Green's function. Since the complex conjugate of both the field as well as the Green's function satisfy differential equations similar to (3) and (7) respectively, i.e.,

$$\nabla^2\phi^* + k_o^2\ \phi^* = -\ \rho^* \tag{95}$$

$$\nabla^2 g^* + k_o^2\ g^* = -\ \delta \quad , \tag{96}$$

it thus follows that the entire development of this report could also have been done in terms of *either* this complex conjugate field *or* this complex conjugate Green's function; with the sole exception that the ansatz would have turned out to be the above holographic ansatz with the (-) sign in the integrand (vis-a-vis the (+) sign in the integrand of (93)). It should be pointed out that in the case of the electromagnetic vector field ansatz (74) and (75), only conjugation of the Green's function yields the current results. Such an approach, although heuristically somewhat attractive, tends, however, to burden the rigorous derivation of the exact inverse formulation and its solution with mathematically cumbersome details, which also obscure the physical meanings involved.

An investigation of the properties, limitations, and physical meaning of this first-order approximation (93) and (94) will be taken up next. To this end, let the source distribution ρ be computed from this first-order approximation; i.e., by the basic wave equation (3),

$$\rho = -\ \nabla^2\phi - k_o^2\ \phi \quad , \tag{97}$$

thus, by (94)

$$\rho = ik_o \, (4\pi)^{-3/2} \, (\nabla^2 + k_o^2) \int_\Omega e^{-i\mathbf{k}_o \cdot \mathbf{x}} \, \psi(k_o) \, d\Omega \tag{98}$$

$$= ik_o \, (4\pi)^{-3/2} \int_\Omega (\nabla^2 + k_o^2) \, e^{-i\mathbf{k}_o \cdot \mathbf{x}} \, \psi(k_o) \, d\Omega \tag{99}$$

$$= ik_o \, (4\pi)^{-3/2} \int_\Omega (-k_o^2 + k_o^2) \, e^{-i\mathbf{k}_o \cdot \mathbf{x}} \, \psi(k_o) \, d\Omega \tag{100}$$

$$= 0 \quad . \tag{101}$$

The same result of (101) can be obtained, although mathematically much more laboriously, by replacing (94) with (93) in (98).

It thus follows from (101) that the source distribution *is not* reconstructable from the first-order solution; and that the fields reconstructed by this first-order solution satisfy the homogeneous free-space wave equation

$$\nabla^2\phi + k_o^2 \, \phi = 0 \quad . \tag{102}$$

This first-order reconstructed source-free field, satisfying the free-space wave equation, is thus consistent with synthetic holography interpretation of the first-order solution (93). The physical meaning of this first-order solution is thus that of first-order scattering (or radiation), and consistent with the physical optics approximation (direct "rays" only).

The reconstruction of the unknown boundary condition - constitutive equation (see sect. VI) which determines the unknown geometrical structure and unknown electromagnetic properties of the scatterer (or radiator) requires knowledge of both the fields and the sources. Thus, since the first-order solution cannot reconstruct the sources, it follows that from an inverse scattering point of view (see sect. VI), that the first-order solution possesses the critical pathology of not being an inverse solution at all.

Examination of the complete solution (73) thus reveals that all the source distribution information, and, hence, all the formal inverse scattering information, is contained only in the resolvent operator of (73); i.e., in all the higher-than-first-order Neumann series solution terms.

SECTION IX

NUMERICAL EVALUATION OF THE RESOLVENT OPERATOR BY MEANS OF THE FAST FOURIER TRANSFORM

Let N be the number of data points in the three-dimensional space $\mathbf{X}$ for which the ansatz $\theta(\mathbf{X})$ has been evaluated by (43). Evaluation of the field $\phi(\mathbf{X})$ for these N points in $\mathbf{X}$ by conventional numerical integration, as implied by the resolvent operator in (73), will thus require N^2 multiply-add operations and storage allocations. To recognize that this is an untenable situation requires merely an examination of the practical order of magnitude of N. A minimally reasonable three-dimensional resolution of the field is of the order of 100 points in each of the three dimensions; i.e., $N=10^6$ and $N^2=10^{12}$, which is totally unacceptable in practice for even the largest and fastest state-of-the-art computers.

In cartesian coordinates, the resolvent operator solution (73) can be written as the three-dimensional convolution equation

$$\phi(\mathbf{X}) = \frac{1}{2}\,\theta(\mathbf{X}) - \frac{k_o^2}{8\pi}\,\frac{1}{|\mathbf{X}|} * \theta(\mathbf{X}) \quad . \tag{103}$$

The resolvent kernel "drops off" with distance as $\frac{1}{r}$; its effect is thus localized to the neighborhood of the field point. The resolvent kernel can thus be truncated at a few wave lengths away from the field point without introducing much of an error, and can certainly be truncated to the complete x-domain $\mathcal{D}(\mathbf{X})$ in which a solution for the field $\phi(\mathbf{X})$ is sought, without introducing much of an error at all. Solution (103) can thus be represented by the *finite* three-dimensional convolution

$$\phi(\mathbf{X}) = \tfrac{1}{2}\,\theta(\mathbf{X}) - \frac{k_o^2}{8\pi} \iiint\limits_{\mathcal{D}(\mathbf{X})} \frac{\theta(\mathbf{X}')}{|\mathbf{X}-\mathbf{X}'|}\, d^3x' \quad . \tag{104}$$

This finite three-dimensional convolution can clearly be evaluated numerically with the aid of the *fast Fourier transform* [22] algorithm applied to a *discrete convolution* [23], for the execution of which the required number of multiply-add operations reduces to $N\ log_2\ N$ and the required number of storage allocations reduces to N. For the previously mentioned practical case of $N=10^6$, $N\ log_2\ N \cong 2\times10^7$, and is thus executable on the CDC 7600 in less than five minutes [24].

For large number of far-field data points $\psi(\mathbf{k}_o)$, it similarly becomes desirable to numerically evaluate the ansatz (43) with the aid of the fast Fourier transform. This is possible since the ansatz integral (43) is essentially a *Fourier-surface-integral* in the observation transform k-space; i.e., since

$$d\Omega = \frac{ds_k}{k^2} \tag{105}$$

where ds_k is the differential k-space surface element, it follows from (43) that

$$\theta(\mathbf{X}) = \frac{1}{(4\pi)^{3/2}\, k_o} \oint_{S_{k_o}} e^{-i\mathbf{k}\cdot\mathbf{X}}\, \psi(\mathbf{k})\, ds_k \tag{106}$$

where the closed surface integration in (106) is over a spherical k-space surface of radius k_o. (The full details of the application of the FFT to (106) will be presented in subsequent report(s) covering the follow-on contract to the contract covered by this report).

SECTION X

FOUR-DIMENSIONAL FORMULATION OF THE INVERSE THEORY

X.1. INTRODUCTION

The objective of the four-dimensional formulation of the inverse problem is to develop an exact inverse theory applicable to time dependent (i.e., moving, rotating, and deforming) radiators and scatterers, utilizing wide-band short-pulse Doppler data. Some aspects of this objective, such as wide-band inverse scattering, can readily be obtained by reformulating the frequency domain inverse solutions into the time domain with the aid of the time-frequency Fourier transform. However, other aspects of this objective, such as the utilization of Doppler data, are at best, extremely difficult to obtain by such means. The source of this difficulty is that all time dependent solutions must be invariant under a Lorentz transformation, which is notoriously cumbersome in three-dimensional formulations (particularly if higher than velocity order effects such as acceleration and jerk are not to be neglected) and remarkably simple in four-dimensional formulations. A four-dimensional relativistically invariant formulation of the inverse problem is thus a most logical approach.

A four-dimensional scalar theory, consistent with the three-dimensional scalar theory developed in the previous sections is presented next. Since the method of derivation of the four-dimensional inverse integral equation and its solution parallels closely the method of derivation of the previously presented three-dimensional theory, this derivation is presented in a brief, but complete, form. However, since again a distribution theory approach is taken, this derivation is commenced with the brief but concise rederivation of the well-known four-dimensional formulation of Stratton [25] of the direct problem, using this distribution theory approach. The purpose of this rederivation is not its elegance, but, as was also the rationale for the presentation of appendix I, its lending credence to the distribution theory approach taken.

X.2. FOUR-DIMENSIONAL FORMULATION OF THE DIRECT PROBLEM

Gauss' theorem applied to the four-dimensional vector field $(g\,\Box\phi - \phi\,\Box g)$ yields

$$\oint_a d\mathbf{a}\cdot(g\,\Box\phi - \phi\,\Box g) = \int_\tau d\tau\;\Box\cdot(g\,\Box\phi - \phi\,\Box g) \quad , \tag{107}$$

where $\mathbf{a}$ is a closed three-dimensional hyper-surface in a four-dimensional hyper-volume τ, where the four-dimensional scalar field ϕ and free-space Green's function g satisfy the inhomogeneous wave equations

$$^2\phi = -\rho \tag{108}$$

$$^2 g = -\delta \tag{109}$$

respectively, and where

$$g = \frac{1}{4\pi^2 r^2} \quad ; \quad \mathbf{r} = \mathbf{x} - \mathbf{x}' \quad , \quad r = |\mathbf{r}| \quad . \tag{110}$$

Thus, by (107) through (109),

$$\oint_a da\cdot(g\,\Box\phi - \phi\,\Box g) = \int_\tau d\tau\ (\Box g\cdot\Box\phi + g\,\Box^2\phi - \Box\phi\cdot\Box g - \phi\,\Box^2 g) \tag{111}$$

$$= \int_\tau d\tau\ (g\,\Box^2\phi - \phi\,\Box^2 g) \tag{112}$$

$$= -\int_\tau d\tau\ g\ \rho\ +\ \int_\tau d\tau\ \phi\ \delta\ . \tag{113}$$

$$\int_\tau d\tau\ \phi\ \delta = \phi'\ ,\ \text{for all } X' \in \tau\ . \tag{114}$$

Thus, by (113) and (114),

$$\boxed{\phi' = \int_\tau d\tau\ g\ \rho\ +\ \oint_a da\cdot(g\,\Box\phi - \phi\,\Box g)\ ,} \tag{115}$$

which is the solution of Stratton [26] for the four-dimensional direct radiation problem.

X.3. FOUR-DIMENSIONAL FORMULATION OF THE INVERSE PROBLEM

Gauss' theorem applied to the four-dimensional vector field $(g\phi)$ yields

$$\oint_a da\cdot\Box(g\phi) = \int_\tau d\tau\ \Box\cdot\Box(g\phi) \tag{116}$$

$$= \int_\tau d\tau\ \Box^2(g\phi)\ , \tag{117}$$

which, by (108) and (109) yields

$$\oint_a d\mathbf{a}\cdot(g\,\Box\phi + \phi\,\Box g) = \int_\tau d\tau\ \Box\cdot(g\,\Box\phi + \phi\,\Box g) \tag{118}$$

$$= \int_\tau d\tau\ (\Box g\cdot\Box\phi + g\,\Box^2\phi + \Box\phi\cdot\Box g + \phi\,\Box^2 g) \tag{119}$$

$$= \int_\tau d\tau\ (2\Box g\cdot\Box\phi - g\,\rho - \phi\,\delta) \quad . \tag{120}$$

If all the non-vanishing source distribution is in τ (i.e., $\rho(\mathbf{X})=0$ for all $\mathbf{X} \notin \tau$), then by (115)

$$\int_\tau d\tau\ g\,\rho = \phi' \quad . \tag{121}$$

Furthermore,,

$$\int_\tau d\tau\ \phi\,\delta = \phi' \quad , \quad \text{for all } \mathbf{X}' \in \tau \quad . \tag{122}$$

Thus, with the aid of (121) and (122), (120) yields

$$\oint_a d\mathbf{a}\cdot(g\,\Box\phi + \phi\,\Box g) = 2\int_\tau d\tau\ \Box g\cdot\Box\phi \ - \ 2\phi' \tag{123}$$

$$\boxed{\phi' - \int_\tau d\tau\ \Box g\cdot\Box\phi = -\tfrac{1}{2}\oint_a d\mathbf{a}\cdot(g\,\Box\phi + \phi\,\Box g) \quad ,} \tag{124}$$

which is the four-dimensional and integro-differential equation representation of the inverse problem.

X.4. SOLUTION OF THE FOUR-DIMENSIONAL INVERSE PROBLEM

Let θ be the known four-dimensional ansatz to the inverse problem, i.e.,

$$\theta \equiv -\tfrac{1}{2} \oint_a d\mathbf{a}\cdot(g\,\Box\phi + \phi\,\Box g) \quad . \tag{125}$$

The far-field ansatz is thus represented by a three-dimensional hypersurface at infinity, and the resulting four-dimensional domain of integration over τ in (124) is thus over all of four-space. For such a far-field ansatz, (124) thus becomes

$$\phi' - \int d\tau\, \Box g\cdot\Box\phi = \theta' \quad , \tag{126}$$

which, by virtue of (110), in four-dimensional cartesian coordinates is

$$\phi(x') - \iiiint_{-\infty}^{\infty} \Box g(x'-x)\cdot\Box\phi(x)\; d^4x = \theta(x') \quad , \tag{127}$$

which can be represented by the four-dimensional convolution equation

$$\phi - \Box g \overset{*}{:} \Box\phi = \theta \quad , \tag{128}$$

The four-dimensional Fourier transform of (128) thus yields (with the aid of the appropriate differentiation rule $\Box \leftrightarrow ik$) in the four-dimensional transform k-space

$$\Phi + k^2\, G\, \Phi = \Theta \quad , \tag{129}$$

where

$$\Phi \leftrightarrow \phi \quad , \quad \Theta \leftrightarrow \theta \quad , \tag{130}$$

$$G \leftrightarrow g \quad . \tag{131}$$

Similarly, (109) yields in four-dimensional k-space

$$G = \frac{1}{k^2} \quad . \tag{132}$$

Combining (129) with (132) thus yields

$$\Phi = \tfrac{1}{2}\,\Theta \quad , \tag{133}$$

which, back in x-space, yields

$$\phi = \tfrac{1}{2}\,\theta \quad , \tag{134}$$

which, with the aid of (125), yields the desired solution

$$\boxed{\phi = -\tfrac{1}{4}\oint d\mathbf{a}\cdot(g\,\Box\phi + \phi\,\Box g) \quad ,} \tag{135}$$

where the domain of integration is a closed three-dimensional hyper-surface at infinity.

It is most remarkable that the resolvent operator of the four-dimensional inverse integral equation (126) is pure local-algebraic (i.e., of the form (134)), vis-a-vis the resolvent operator of the three-dimensional inverse integral equation (25), which is a global integral operator (i.e., of the form (73)). This simplicity, which occurs in four-dimensional space only, is intimately connected with another remarkable property unique to four-dimensional space; namely, that the *functional form* of the Green's function in k-space is *invariant* to the *dimensionality* of the space, which is *not* the case in x-space; and, that *only* in four-dimensional space, is the functional form of the Green's function the *same* both in x- *and* k-space [27]; i.e.,

$$G(\mathbf{k}) = \frac{1}{k^2} \leftrightarrow \frac{1}{r^2} = g(\mathbf{x}) \tag{136}$$

in four-dimensional space only.

X.5. CONCLUDING REMARKS

The extension of the preceding four-dimensional scalar field inverse theory to vector and tensor fields, appropriately applicable to electromagnetic fields, can be accomplished by means somewhat analogous to those of sect. V.

The Lorentz-invariant four-dimensional reformulation of Maxwell's equations for the total charge and current densities in terms of the fundamental electromagnetic fields (see appendix II, eqn. (32) - (35)), which is needed for the previously mentioned extension of the inverse theory, can be accomplished by most conventional means.

A four-dimensional Lorentz-invariant generalized boundary condition - constitutive equation (see appendix II, eqn. (38) and (49)), which is essential to the final objectives of a four-dimensional inverse theory (see sect. VI) is similarly obtainable.

The various permutations and combinations of the special case solutions (and their details) for wide-band short-pulse data, monostatic-bistatic data, Doppler data, and incomplete ansatz data, etc., can be extracted from the general solution by an appropriate choice of the four-dimensional geometry, the coordinate system, and the domain of hyper-surface integration implied by the ansatz (125) and the general solution (135); i.e., by an appropriate choice of the "world-line" of the observer relative to the radiator or scatterer, or vice-versa.

The work outlined in the preceding four paragraphs is currently being undertaken under an appropriate part of a follow-on contract to the contract covered by this report, and will be covered in detail in subsequent report(s) to this follow-on contract.

Since, as was the case with the three-dimensional inverse solution (see sect. IX), the four-dimensional inverse solution (135) can be formulated as a convolution; it can thus similarly be evaluated numerically with the aid of the fast Fourier transform applied to its equivalent discrete convolution representation. However, since such a convolution is four-dimensional, the size of practical data (e.g., $(64)^4 = 1.7\times10^7$) will utilize to the limit the size and speed of the biggest and fastest existing computer [28]. Next generation computers, such as Star or Illiac, will not be taxed to the limit of their size and speed by four-dimensional problems of practical magnitude. An alternative practical means of evaluating (135) is by a hard-wired *Fast Fourier Analyzer* (i.e., by special purpose computer, vis-a-vis the software compiled FFT on a general purpose computer). A further alternative might be a Fourier analog optical processor.

In previous attempts at a formal definition of the inverse problem (see sect. VI), the seemingly logical classification of the direct vs. the inverse problem in terms of the unknown being the *effect* vs. the *cause* respectively, has purposely been avoided for reasons that will be discussed next. The topological distinction between the three-dimensional direct and inverse problem has previously been shown to be that of whether the unknown fields and sources are on different or the same side of the closed surface ansatz integrals $\oint d\mathbf{S}\cdot(g\ \nabla\phi \mp \phi\ \nabla g)$ respectively. This topological distinction clearly carries over into four-dimensions for the closed hyper-surface ansatz integral $\oint d\mathbf{a}\cdot(g\Box\phi \mp \phi\Box g)$. However, because of the temporal nature of the added fourth dimension and the (lorentz transformation invariant) causality principle, it is now clear that no Lorentz-invariant distinction between the direct and inverse problems can be made on the basis of cause and effect.

APPENDIX I

SIMPLIFIED AND UNIFIED REDERIVATION OF THE INTEGRATION OF THE FIELD EQUATIONS FOR THE DIRECT PROBLEM

Gauss' theorem, when applied to the vector field ($\phi \nabla g - g \nabla\phi$), yields

$$\oint_s d\mathbf{s}\cdot(\phi \nabla g - g \nabla\phi) = \int_v dv\ \nabla\cdot(\phi \nabla g - g \nabla\phi) \quad , \tag{1}$$

provided ($\phi \nabla g - g \nabla\phi$) is continuous and twice-differentiable in v and on s. These continuity conditions can be totally dispensed with if ϕ and g and, hence, ($\phi \nabla g - g \nabla\phi$), are taken as distributions [29]; specifically, if ϕ is taken as a field satisfying the inhomogeneous Helmholtz equation

$$\nabla^2\phi(\mathbf{x}) + k^2\ \phi(\mathbf{x}) = -\ \rho(\mathbf{x}) \quad , \tag{2}$$

and g is taken as the associated Green's function

$$g(\mathbf{x}|\mathbf{x}') = \frac{e^{ikr}}{4\pi r} \quad , \tag{3}$$

$$\mathbf{r} = \mathbf{x} - \mathbf{x}' \ , \tag{4}$$

$$r = |\mathbf{r}| \quad , \tag{5}$$

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which satisfies the inhomogeneous differential equation

$$\nabla^2 g(\mathbf{x}) + k^2\, g(\mathbf{x}) = -\,\delta(\mathbf{x}) \quad , \tag{6}$$

and where both ϕ and g satisfy the radiation condition at infinity.

The right-hand side volume integral in (1) reduces to

$$\int_V dv\ \nabla\cdot(\phi\ \nabla g - g\ \nabla\phi) = \int_V dv\ (\nabla\phi\cdot\nabla g + \phi\ \nabla^2 g - \nabla g\cdot\nabla\phi - g\ \nabla^2\phi) \tag{7}$$

$$= \int_V dv\ (\phi\ \nabla^2 g - g\ \nabla^2\phi) \quad , \tag{8}$$

which thus yields for (1)

$$\oint_S d\mathbf{S}\cdot(\phi\ \nabla g - g\ \nabla\phi) = \int_V dv\ (\phi\ \nabla^2 g - g\ \nabla^2\phi) \quad , \tag{9}$$

which is known as Green's second identity in Green's theorem.

With the aid of (2) and (6), (9) reduces to

$$\oint_S d\mathbf{S}\cdot(\phi\ \nabla g - g\ \nabla\phi) = \int_V dv\ [\phi(-k^2 g-\delta) - g(-k^2\phi-\rho)] \tag{10}$$

$$= \int_V dv\ (-k^2 g\phi - \delta\phi + k^2 g\phi + g\rho) \tag{11}$$

$$= \int_V dv\ (-\delta\phi + g\rho) \tag{12}$$

$$\oint_s d\mathbf{S}\cdot(\phi\ \nabla g - g\ \nabla\phi) = -\int_v dv\ \delta\ \phi\ + \int_v dv\ g\ \rho \tag{13}$$

If the field point X' is taken as inside the volume v bound by the surface s, then the first volume integram on the right-hand side of (13) yields by the very definition of the delta-function

$$\int_v dv\ \delta(X-X')\ \phi(X) = \phi(X') \quad ; \quad X' \in v\ . \tag{14}$$

Similarly, if the field point X' is taken as outside this volume v, then

$$\int_v dv\ \delta(X-X')\ \phi(X) = 0 \qquad ; \quad X' \notin v\ . \tag{15}$$

And similarly again, if the field point X' is on the surface s bounding the volume v, then

$$\int_v dv\ \delta(X-X')\ \phi(X) = \frac{\Omega(X')}{4\pi}\ \phi(X') \quad ; \quad X' \in s\ , \tag{16}$$

where $\Omega(X')$ is the internal solid angle subtended by s on X', Furthermore, if the curvature of s at X' is finite and non-singular, then $\Omega(X') = 2\pi$; then

$$\int_v dv\ \delta(X-X')\ \phi(X) = \tfrac{1}{2}\ \phi(X') \qquad . \tag{17}$$

If the internal solid angle $\Omega(X')$ subtended by s on X' is consistently generalized for the field point X' *not* on s as

$$\Omega(\mathbf{X}') \equiv \begin{cases} 4\pi \ , & \mathbf{X}' \in v \\ 0 \ , & \mathbf{X}' \notin v \ ; \end{cases} \tag{18}$$

which is geometrically most reasonable, (14) through (16) are representable by

$$\int_v dv \ \delta(\mathbf{X}-\mathbf{X}') \ \phi(\mathbf{X}) = \frac{\Omega(\mathbf{X}')}{4\pi} \phi(\mathbf{X}') \quad \text{for all } \mathbf{X}'. \tag{19}$$

With the aid of this (19), (13) yields

$$\boxed{\frac{\Omega'}{4\pi} \phi' = \int_v dv \ g \ \rho \ + \oint_s d\mathbf{s}\cdot(g \ \nabla\phi - \phi \ \nabla g) \ ,} \tag{20}$$

which is a unified representation of the Kirchhoff integration of the field equations, valid for all field points (i.e., field points inside, on, and outside s).

A physical interpretation, as well as a rigorous proof of the physical meaning of both the volume and the surface integrals in (20) are of interest next. To this end, consider a closed surface s_1 totally inside another closed surface s_2 (see fig. 12), a source distribution $\rho_1(\mathbf{X})$ totally localized in the volume v_1 bound by the closed surface s_1 (i.e., $\rho_1(\mathbf{X}) = 0$ for all $\mathbf{X}$ outside s_1), and another source distribution ρ_2 (distinct from ρ_1) totally localized in the intervening volume v_2 band by the closed surfaces s_1 and s_2 (i.e., $\rho_2(\mathbf{X}) = 0$ for all $\mathbf{X}$ inside s_1 and outside s_2). Furthermore, let the field point $\mathbf{X}'$ be outside the closed surface s_1 and inside the closed surface s_2; i.e., in the volume v_2. It thus follows that if the closed surface s in (20) is taken as the closed surface s_1 and s_2, then by (18), (20) yields

$$\phi' = \int_{v_2} dv \ g \ \rho_2 + \oint_{s_1} d\mathbf{s}\cdot(g \ \nabla\phi - \phi \ \nabla g) + \oint_{s_2} d\mathbf{s}\cdot(g \ \nabla\phi - \phi \ \nabla g) \ . \tag{21}$$

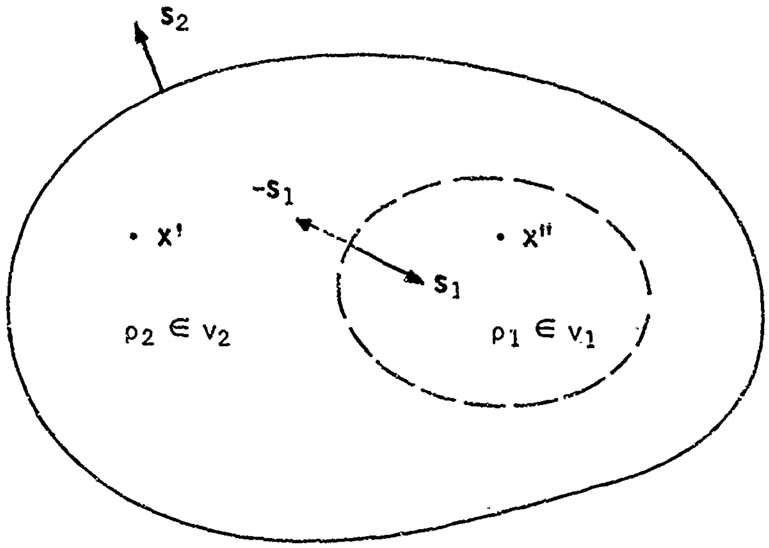

FIGURE 12
FIELD AND SOURCE POINT GEOMETRY

If, however, the closed surface s in (20) is taken as the closed surface s_2 only, then by (18), (20) yields

$$\phi' = \int_{v_1} dv\ g\ \rho_1 + \int_{v_2} dv\ g\ \rho_2 + \oint_{s_2} d\mathbf{S}\cdot(g\ \nabla\phi - \phi\ \nabla g) \quad . \tag{22}$$

Examination of (21) and (22) reveals that the sole distinction between these two representations of the field at the field point $\mathbf{X}'$ is the interchangeability of the closed surface integral over the closed surface s_1 with the volume integral over the volume v_1. It thus follows that the field ϕ_1 at the field point $\mathbf{X}'$, due solely to the source distribution ρ_1, can be represented by either the volume integral

$$\phi_1' = \int_{V_1} dv\ g\ \rho_1 \tag{23}$$

or the closed surface integral

$$\phi_1' = \oint_{S_1} d\mathbf{S}\cdot(g\ \nabla\phi_1 - \phi_1\ \nabla g) \quad . \tag{24}$$

That these volume and surface integrals (23) and (24) are indeed equivalent can also be shown by taking the closed surface s in (20) as the closed surface s_1 only; then, by (18), (20) yields

$$0 = \int_{V_1} dv\ g\ \rho_1 \ - \oint_{S_1} d\mathbf{S}\cdot(g\ \nabla\phi_1 - \phi_1\ \nabla g) \quad ; \quad \textit{Q.E.D.;} \tag{25}$$

(where the (-) sign in front of the surface integral in (25) is due to the fact that by Gauss' theorem the surface unit vector $\hat{s}$ must always be chosen as outwards from the domain of volume integration; see fig. 12).

The physical meanings and interpretations of (23) and (24) respectively are clearly that of the *sum* (volume integral) *of the action-at-a-distance-field due to a spatial* (volume) *source distribution* and its *closed surface integral equivalent*, provided the field point X' and the source distribution ρ_1 are outside and inside of the closed surface s_1 respectively.

Next, consider the field point X" chosen as inside the closed surface s_1 (see fig. 12); thus, if the closed surface s in (20) is taken as the closed surface s_1, then by (18), (20) yields for the field ϕ'' at this field point X"

$$\phi'' = \int_{V_1} dv\ g\ \rho_1 \ - \oint_{S_1} d\mathbf{S}\cdot(g\ \nabla\phi - \phi\ \nabla g) \quad ; \tag{26}$$

(where the (-) sign in front of the surface integral in (26) is again for the same reasons as stated subsequent to (25)).

If, however, the closed surface s in (20) is taken as the closed surface s_1 and s_2, then by (18), (20) yields (for the field point still chosen as X")

$$0 = \int_{V_2} dv\ g\ \rho_2 + \oint_{S_1} d\mathbf{S}\cdot(g\ \nabla\phi - \phi\ \nabla g) \quad . \tag{27}$$

It thus follows from the previously developed physical meaning and interpretation of the volume integral over a source distribution, and comparison of (26) and (27), that the physical meaning and interpretation of the closed surface integral in (26) is that of the (negative of the) *equivalent of the action-at-a-distance-field due to a spatial source distribution*, provided the field point X" and the source distribution ρ_2 are inside and outside of the closed surface s_1 respectively (the (-) sign again having arisen for the previously stated reasons).

Since in the previous arguments the closed surface, the field point, and the nonvanishing source distribution, were chosen topologically completely arbitrarily, with the sole exception that the field point and the nonvanishing source distribution are to be on *opposite sides* of the closed surface, it follows that if the surface unit vector is consistently chosen as pointing from the field point to the source distribution, then the physical meaning and interpretation of the surface integral $\oint d\mathbf{S}\cdot(g\ \nabla\phi - \phi\ \nabla g)$ is *always* that of the *fields produced at a field point on ONE SIDE of the surface* by *the total source distribution located on the OTHER SIDE of the surface*, and totally equivalent to the volume integral $\int g\ \rho\ dv$ over this *same* source distribution.

This property of relating fields and sources on *opposite* sides of the Kirchhoff surface integral is its fundamental topological property, when applied to *direct* field problems; it is, however, totally useless when applied to inverse field problems; since the latter problem is topologically characterized by unknown fields and sources on the *same* side of a closed surface. The surface integral of interest to the *inverse* field problem is thus a surface integral relating fields and sources on the *same* side of the surface; relating them at least in such a fashion as to render the surface integral a useful ansatz to the inverse problem. As shown in sect. II, such a surface integral is $\oint d\mathbf{S}\cdot(g\ \nabla\phi + \phi\ \nabla g)$.

Replacing the scalar field Gauss' theorem (1) with its vector equivalent

$$\oint_S d\mathbf{S}\times(\nabla g\times\mathbf{F} - g\ \nabla\times\mathbf{F}) = \int_V dv\ \nabla\times(\nabla g\times\mathbf{F} - g\ \nabla\times\mathbf{F}) \quad , \tag{28}$$

and applying it analogously twice to the electromagnetic vector field and wave equations (see appendix II), vis-a-vis the scalar Helmholtz field equation (2), where the vector field $\mathbf{F}$ represents the electric and magnetic vector fields respectively, similarly yields the familiar Stratton and Chu [30] or Franz [31] integral representation of Maxwell's equations.

As in the case of the scalar field, the surface integral in (28) relates fields and current density distributions (the source distributions) of opposite sides of this surface; whereas the surface integral $\oint d\mathbf{S}\times\nabla\times(g\mathbf{F})$ relates fields and current densities which are on the same side of the surface.

APPENDIX II

A REFORMULATION OF THE ELECTROMAGNETIC FIELD EQUATIONS

II.1. THE CONVENTIONAL FREE CHARGE AND CURRENT DENSITIES REPRESENTATION

The conventional representation of Maxwell's equations is

$$\nabla \cdot \mathbf{D} = \rho_f \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \quad , \tag{4}$$

where ρ_f and $\mathbf{J}_f$ are the free charge and current densities respectively (also referred to by some authors as the true or unbound charge and current densities); $\mathbf{E}$ and $\mathbf{B}$ are the electric and induction fields respectively, which are fundamental force fields; and $\mathbf{D}$ and $\mathbf{H}$ are the displacement and magnetic fields respectively, which are artificially derived fields. These artificially derived fields are related to the fundamental fields by

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}_b \tag{5}$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad , \tag{6}$$

where $\mathbf{P}_b$ and $\mathbf{M}$ are the polarization (of the bound charges) and magnetization fields respectively. These polarization and magnetization fields are the contributions to the fundamental fields produced by local charge and current densities in a material medium. ε_0 and μ_0 are the permittivity and permeability of free space respectively, collectively referred to as the inductive capacities of free space.

In linear isotropic media, these polarization and magnetization fields are relatable to the electric and magnetic fields respectively by the constitutive equations

$$P_b = \chi_e \varepsilon_0 E \tag{7}$$

$$M = \chi_m H \quad , \tag{8}$$

where χ_e and χ_m are the electric and magnetic susceptibilities of the linear isotropic medium respectively, collectively referred to as the susceptibilities of the medium.

For such linear isotropic media, with the aid of (5) through (8), the relationship between the derived and fundamental fields yields the conventional constitutive equations

$$D = \varepsilon E \tag{9}$$

$$B = \mu H \quad , \tag{10}$$

where

$$\varepsilon = \kappa_e \varepsilon_0 \tag{11}$$

$$\kappa_e = 1 + \chi_e \quad , \tag{12}$$

and

$$\mu = \kappa_m \mu_0 \tag{13}$$

$$\chi_m = 1 + \chi_m \quad , \tag{14}$$

where ε and μ are the permittivity and permeability of the medium respectively, collectively referred to as the inductive capacities of the medium; and κ_e and κ_m are the relative permittivity (or dielectric constant) and relative permeability of the medium respectively, collectively referred to as the specific inductible capacities of the medium.

For linear isotropic conducting media, Ohm's law yields the additional constitutive equation

$$J_f = \sigma_f E \quad , \tag{15}$$

where σ_f is the conductivity (of free charges of the medium.

For linear isotropic media which are also spatially and temporally homogeneous (i.e., media for which the inductive capacities are constants neither a function of space nor a function of time), the conventional formulation of the wave equations follows from (3), (4), (9), and (10); i.e.,

$$\nabla\times\nabla\times E + \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = -\mu \frac{\partial J_f}{\partial t} \tag{16}$$

$$\nabla\times\nabla\times H + \mu\varepsilon \frac{\partial^2 H}{\partial t^2} = \nabla\times J_f \quad . \tag{17}$$

For the general case of inhomogeneous media (i.e., media for which either or both of the inductive capacities are not a constant; namely, are a function of time and/or space), the formulation of such a wave equation is in general *not* possible.

The preceding equations can be classified into one of three classes:

MAXWELL'S EQUATIONS: The first order differential equations (1) through (4) relating the electric and magnetic fields to the charge and current densities. In these equations the electric and magnetic fields are intertwined and not separated. These equations are the fundamental equations derivable from the fundamental action-at-a-distance laws of Coulomb, Faraday, and Biot-Savart.

THE WAVE EQUATIONS: The second order differential equations (16), (17) relating the electric or magnetic fields separately to the current densities.. The essential and distinguishing (from Maxwell's equations) property of these equations is that the electric and magnetic fields are separated, i.e., that each wave equation contains one field (electric or magnetic) only. The term *wave equation* is thus possibly a misleading misnomer; a more characteristically descriptive name might thus be the *separated equations*. It should again be noted that for the general case of inhomogeneous media, the separation of the electric and magnetic fields into a set of wave equations of the form (16) and (17) cannot be achieved.

Maxwell's equations and the wave equations, namely, all those equations derivable from the fundamental action at a distance laws, are collectively referred to as the *field equations*.

THE CONSTITUTIVE EQUATIONS: The local algebraic equations (7), (8), (9), (10), and (15) which relate the fundamental fields to the artificially derived fields, or the current densities to the fields. These equations are a description of the local properties of the material medium, such as Ohm's law, and are essentially non-electromagnetic in nature (namely, they do not contain the basic electromagnetic action-at-a-distance laws); in most cases, these equations involve Newton's second law of motion and the microscopic dynamics of the material medium.

It takes a (set of) field equations *and* a (set of) constitutive equations to completely specify any electromagnetic problem. (The constitutive equations often take the form of a boundary condition to the differential field equations; the boundary conditions determining the specific unique solution to the differential equations).

II.2. THE DIFFICULTIES WITH THE CONVENTIONAL FORMULATION

The difficulty with the conventional formulation of the field equations is essentially the severely restricted applicability of the resulting wave equation to the interior, and not the surface, of homogeneous media only.

It thus immediately follows that the conventional formulation of the integral representation of the field equations is also valid for the interior, and not the surface, of homogeneous media only; i.e., the complete domain of volume and surface integration must be in the interior of the homogeneous medium.

The seriousness of this limitation can best be illustrated by the case of a finite sized homogeneous medium of constant inductive capacities different from those of empty space, imbedded in infinite free space. For such a case, one clearly obtains two different wave equations (of the form of (16) and (17), one valid inside the medium and one valid outside of the medium, neither of which is valid on the surface between the medium and the free space. Furthermore, there exists no single wave equation valid for all space. The conventional formulation of the integral representation of the field equations is thus valid only if the complete domain of volume and surface integration is completely in the interior of such a medium; this integral representation is thus not valid for such problems as the external scattering by such media.

For the general case of arbitrarily inhomogeneous media, a wave equation, though much more complicated than the form (16) and (17) is formable; however, no appropriate Green's function is known for such a wave equation. Without such a Green's function, the integration of the field equations into an integral representation is thus not possible.

II.3. THE TOTAL CHARGE AND CURRENT DENSITIES REPRESENTATION

The source of the difficulties with the conventional formulation of the field equations lies in their provincial view of matter, charge, and current, and, hence, the fields. Maxwell's equations contain only the free charge and current densities, hence, the artificially derived fields in addition to the fundamental fields. The presence of the artificially derived fields in Maxwell's equations furthermore injects into the field equations some of the properties of the medium, which rightfully should be in a separate set of constitutive equations. The distinction between free and bound charge and current densities is artificial and a source of considerable difficulty when dealing with *non-classical, non-linear* media such as plasmas and semiconductors. Furthermore, all accelerating charges, whether free or bound, radiate electromagnetic fields. It thus stands to reason that a more fundamental formulation of Maxwell's equations would contain only the *total* charge and current densities and only the fundamental fields, and thus be free of the artificially derived fields and the artificial distinction between free and bound charge and current densities. The properties of the medium implicit in the artificially derived fields should then only appear in a separate set of constitutive equations. Such reasoning is further enhanced by the point of view of an observer of electromagnetic radiation, to whom in principle no distinction exists between electromagnetic radiation emanating from accelerating free or bound charges, and who measures such radiated fundamental fields at an observation point with the aid of the properties of the medium in which the observation point is imbedded.

The sought reformulation of Maxwell's equations is thus a formulation *invariant* to the specific properties of the medium and the mechanistic nature of the free and bound charges in the medium; namely, a formulation stating the *universal* laws of the electromagnetic theory *only*. All the specific properties of the medium would thus be relegated solely to a set of constitutive equations.

Such an invariant formulation of Maxwell's equations can be derived directly from the basic *action-at-a-distance* laws of Coulomb, Faraday, and Biot-Savart, and an argument analogous to Maxwell's argument that yielded the displacement current, utilizing an equation of continuity for the conservation of the total charges only. However, for the sake of showing consistency with the conventional formulation of Maxwell's equations, such an invariant formulation will be developed next directly from the conventional formulation and the definitions of the artificially derived fields.

Eliminating the artificially derived fields from (1) through (4) with the aid of (5) and (6) yields

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\,(\rho_f - \nabla\cdot\mathbf{P}_b) \tag{18}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{19}$$

$$\nabla\times\mathbf{E} = -\,\frac{\partial\mathbf{B}}{\partial t} \tag{20}$$

$$\nabla\times\mathbf{B} = \mu_0\,\left(\mathbf{J}_f + \frac{\partial\mathbf{P}_b}{\partial t} + \nabla\times\mathbf{M}\right) + \mu_0\varepsilon_0\,\frac{\partial\mathbf{E}}{\partial t}\quad . \tag{21}$$

From the very definitions of the polarization and magnetization fields associated with the material media, it follows that

$$\nabla\cdot\mathbf{P}_b = -\,\rho_f \tag{22}$$

$$\frac{\partial\mathbf{P}_b}{\partial t} = \mathbf{J}_p \tag{23}$$

$$\nabla\times\mathbf{M} = \mathbf{J}_m \quad , \tag{24}$$

where ρ_b is the bound polarization charge density; namely, the net bound charge density produced by the polarization field in a material medium; and $\mathbf{J}_p$ and $\mathbf{J}_m$ are the bound polarization and magnetization current densities respectively; namely, the net bound current densities produced by the polarization and magnetization fields respectively in a material medium.

Since the total charge and current densities ρ and $\mathbf{J}$ respectively is the sum of all the free and bound charge and current densities respectively, irrespective of the mechanistic nature of these bound charge and current densities, it follows that

$$\rho = \rho_f - \nabla\cdot\mathbf{P}_b \tag{25}$$

$$\mathbf{J} = \mathbf{J}_f + \frac{\partial \mathbf{P}_b}{\partial t} + \nabla\times\mathbf{M} \quad . \tag{26}$$

The invariant formulation of Maxwell's equations in terms of the total charge and current densities thus becomes

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_o}\,\rho \tag{27}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{28}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{29}$$

$$\nabla\times\mathbf{B} = \mu_o\,\mathbf{J} + \mu_o\varepsilon_o\,\frac{\partial \mathbf{E}}{\partial t} \quad . \tag{30}$$

It now becomes convenient to redefine the magnetic field $\mathbf{H}$ as a fundamental field, related to the fundamental induction field $\mathbf{B}$ for all media as

$$\mathbf{H} \equiv \frac{1}{\mu_o}\,\mathbf{B} \quad . \tag{31}$$

Maxwell's equations (27) through (30), in terms of this redefined fundamental magnetic field thus become

$$\boxed{\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{1}{\varepsilon_o}\,\rho & (32)\\ \nabla\cdot\mathbf{H} &= 0 & (33)\\ \nabla\times\mathbf{E} &= -\mu_o\,\frac{\partial \mathbf{H}}{\partial t} & (34)\\ \nabla\times\mathbf{H} &= \mathbf{J} + \varepsilon_o\,\frac{\partial \mathbf{E}}{\partial t} \quad . & (35) \end{aligned}}$$

The invariant formulation of the wave equations, no longer restricted to linear, isotropic, and homogeneous media, but *universally* valid for all media, now becomes

$$\boxed{\begin{aligned} &\nabla\times\nabla\times\mathbf{E} + \mu_0\varepsilon_0 \frac{\partial^2\mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial\mathbf{J}}{\partial t} \\ &\nabla\times\nabla\times\mathbf{H} + \mu_0\varepsilon_0 \frac{\partial^2\mathbf{H}}{\partial t^2} = \nabla\times\mathbf{J} \quad , \end{aligned}} \qquad \begin{aligned} (36) \\ (37) \end{aligned}$$

The three constitutive equations (7), (8), and (15), with the aid of (23) and (24), thus immediately yield the *single* constitutive equation for the total current density

$$\mathbf{J} = \sigma_f \mathbf{E} + \varepsilon_0 \frac{\partial}{\partial t}(\chi_e \mathbf{E}) + \nabla\times(\chi_m \mathbf{H}) \quad . \tag{38}$$

The boundary condition for the surface of perfect conductors, i.e.,

$$\mathbf{K} = \mathbf{n}\times\mathbf{H} \quad , \tag{39}$$

where $\mathbf{K}$ is the surface current density and $\mathbf{n}$ is the outward surface unit vector, presents a special problem that can be resolved by introducing the magnetic vector surface conductivity distribution density $(\frac{d\mathbf{s}}{dv})$ such that, since $\mathbf{J}\, dv = \mathbf{K}\, ds$, the boundary condition (39) becomes the constitutive (distribution) equation

$$\mathbf{J} = \frac{d\mathbf{s}}{dv} \times \mathbf{H} \quad . \tag{40}$$

With the aid of (38), it is thus possible to introduce the *generalized constitutive equation*

$$\boxed{\mathbf{J} = \sigma_f \mathbf{E} + \varepsilon_0 \frac{\partial}{\partial t}(\chi_e \mathbf{E}) + \chi_m \nabla\times\mathbf{H} + (\nabla\chi_m + \frac{d\mathbf{s}}{dv})\times\mathbf{H} \quad .} \tag{41}$$

The equation of continuity for the conservation of the total charges thus is

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0 \quad . \tag{42}$$

The time-reduced form of Maxwell's equations (32) through (35) for monochromatic fields in terms of the total charge and current density thus is

$$\nabla \cdot E = \frac{1}{\varepsilon_o} \rho \tag{43}$$

$$\nabla \cdot H = 0 \tag{44}$$

$$\nabla \times E = - i\omega\mu_o H \tag{45}$$

$$\nabla \times H = J + i\omega\varepsilon_o E \quad . \tag{46}$$

With the aid of (44), the time-reduced form of the wave equations (36) and (37) for monochromatic fields in terms of the total current density thus is

$$\nabla \times \nabla \times E - k^2 E = - i\omega\mu_o J \tag{47}$$

$$\nabla^2 H + k^2 H = - \nabla \times J \quad . \tag{48}$$

The time-reduced form of the generalized constitutive equation (41) for monochromatic fields for the total current density thus is

$$J = (\sigma_f + i\omega\varepsilon_o \chi_e)E + (\nabla\chi_m + \chi_m \nabla + \frac{dS}{dv}) \times H \quad ; \tag{49}$$

And the time-reduced form of the equation of continuity for the conservation of total charges thus is

$$\nabla \cdot J = - i\omega \rho \quad . \tag{50}$$

A universally valid scalar, vector, and Hertz potential theory for the fundamental fields only, in terms of the total charge and current densities, can similarly and consistently be developed.

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13. ABSTRACT

PART I: A three-diemsnional electromagnetic Inverse Scattering Identity, based on the Physical Optics approximation, is developed for the monostatic scattered far field cross section of perfect conductors. Uniqueness of this inverse scattering identity is proven. This identity requires complete scattering information for all frequencies and aspect angles. A non-singular integral equation is developed for the arbitrary case of incomplete frequency and/or aspect angle scattering information. A general closed form solution to this integral equation is developed, which yields the shape of a scatterer from such incomplete information. A specific practical radar solution is presented. The resolution of this solution is developed; yielding short-pulse target resolution radar system parameter equations. Results of the three-dimensional numerical reconstruction of a sphere and a cylinder from a variety of aspect angle and frequency band limited cross section data are presented. The merits of this solution over the conventional synthetic aperture radar imaging technique are discussed.

PART II: The general inverse scattering and radiation problem associated with the three-dimensional inhomogeneous scalar field Helmholtz wave equation is formulated as a Fredholm integro-differential equation of the second kind. The far-field inverse integro-differential equation is solved in closed form with the aid of a single resolvent integral operator, which can be readily evaluated numerically with the aid of the Fast Fourier transform algorithm. The inverse integro-differential equation and its solution are then generalized to the reduced vector wave equation resulting from Maxwell's equations. A formal statement of the inverse problem is presented. It is shown that the first order Neumann series solution of the inverse integro-differential equation as well as the first order term of its exact solution represent the physical optics approximation and the equations governing synthetic microwave holography. The (analogous) four-dimensional inverse integro-differential equation and its closed form solution, applicable to Doppler-contaminated fields radiated by moving scatterers, is developed.

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